ALGEBRA FOR BEGINNERS

Minerva E. Leland

Bradbury's Mathematical Series.

ALGEBRA FOR BEGINNERS.

BY

WILLIAM F. BRADBURY, A.M.,

HEAD MASTER OF THE CAMBRIDGE LATIN SCHOOL,

AND

GRENVILLE C. EMERY, A.M.,

MASTER IN THE BOSTON LATIN SCHOOL.

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PREFACE.

THIS work has been prepared expressly for beginners, and in response to a call for an Algebra for the higher classes in Grammar Schools.

It is taken for granted that the pupil is familiar with the principles of ordinary Arithmetic.

Few rules and definitions are given; the use of algebraic language is illustrated by numerous exercises; and the elementary principles of Algebra are made clear by the introduction of easy problems.

In order to awaken the interest of the pupil, the Equation, its reduction, and numerous problems are introduced at the very beginning of the book.

The examples given are carefully graded from the very simple to the more difficult. A considerable part of them have been tested by teachers of the grade for which they are designed. After the earlier problems, care has been taken to introduce for the most part such as require algebraic principles in their solution rather than those that can be more easily and naturally solved without the aid of Algebra. There is nothing better for training the intellectual powers than putting into algebraic language the conditions of a mathematical problem.

The subjects introduced and the method of treatment are such as to give the pupil a substantial ground-work for the more advanced work in Algebra.

It is not essential that every example and problem given in any topic should be done before the pupil passes to the succeeding subject. The teacher should use his judgment in each case, varying the number of exercises taken according to the ability of the pupil and the time allowed for the study. This is especially true in the subject of Factoring, of the Greatest Common Divisor of Polynomials, and of the Least Common Multiple of Polynomials. Further, it is not essential to follow in every respect the order of topics given in the book. If the teacher does not approve of the plan, adopted by the authors, of beginning with the Equation, he can defer the whole or any part of the first four chapters till a part or the whole of Chapters V.–XIV. is completed.

Some of the miscellaneous equations and problems at the end of the book are a little more difficult than those in the body of the work, and are to be selected for use at the discretion of the teacher.

W. F. B.

G. C. E.

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ALGEBRA FOR BEGINNERS.

CHAPTER I.

(SEE PREFACE.)

1. Mr. Hardy had three sons, Francis, Ralph, and John. On one occasion he distributed among them some marbles, giving to Francis, the eldest, twice as many as to Ralph, and to Ralph twice as many as to John, the youngest.

If now we are told that he gave John 2 marbles, we at once know that Ralph received 4, and Francis 8.

Suppose now we let n stand for the number of marbles John receives: then $2n$ will stand for the number Ralph receives; and $4n$ for the number Francis receives; and n and $2n$ and $4n$, or $7n$, will be the whole number of marbles given to the three boys.

n and $2n$ and $4n$ equal $7n$.

For *and* we generally use in Algebra *plus*.

n plus $2n$ plus $4n$ equals $7n$.

2. About the year 1550 A. D., Stifel, a German, suggested the use of the sign + for the word *plus;* and shortly after this, Robert Recorde, an Englishman, invented the sign =, for *equals*.

For the statement above, then, we write

$$n + 2n + 4n = 7n$$

3. It is easily seen that the number of marbles the father gives, less the number he gives the eldest son Francis, is equal to the number he gives to Ralph and John; or

$$7\,n \text{ less } 4\,n = 2\,n + n$$

For *less* we generally use in Algebra *minus.*

$$7\,n \text{ minus } 4\,n = 2\,n + n$$

For the word *minus* the same Stifel suggested the sign —, and we now write

$$7\,n - 4\,n = 2\,n + n$$

4. For the sake of brevity, signs were invented for use not only in addition and subtraction, but also in multiplication and division.

Thus for *multiplied by* we use the sign $\times$, or the dot $\cdot$ above the line; thus, 7×5, or $7 \cdot 5$, means that 7 is to be multiplied by 5. Between a figure and a letter, and between letters, the sign of multiplication is usually omitted; thus $7\,n$ means $7 \times n$. But when figures are to be multiplied together the sign of multiplication cannot be omitted; thus 35 does not mean 3×5 but 3 tens and 5 units, or thirty-five. For *divided by* we use the sign $\div$, or $:$; thus, $8 \div 4$, or $8 : 4$, means that 8 is to be divided by 4. Division is also expressed by the fractional form $\frac{8}{4}$. All these expressions for division have for their value 2.

5. The sign $\therefore$ is often used instead of the word *hence,* or *therefore.*

6. The expression $n + 2\,n + 4\,n = 7\,n$ is called an *equation.* The parts to the left and right of the sign $=$ are called *members,* or *sides,* and are distinguished as the *first member* and *second member,* or the *left side* and *right side.*

7. An equation, and the processes used in its reduction, can be illustrated by a lever, or see-saw, with equal weights at equal distances from the fulcrum, or balancing point.

Arthur + Bertha = Caroline + David

A + B = C + D

8. It is evident in order that the lever may remain in its horizontal position that Arthur and Bertha must together weigh the same as Caroline and David; and if the weight on either end is changed in any way, the weight on the other end must be changed in like amount. So, *in an equation, whatever is done to one side must be done to the other*, in order that the equality may remain. That is,

1. If anything is added to one side, an equal amount must be added to the other.

2. If anything is subtracted from one side, an equal amount must be subtracted from the other.

3. If one side is multiplied by any number, the other side must be multiplied by an equal number.

4. If one side is divided by any number, the other side must be divided by an equal number.

Such self-evident statements are called *Axioms*.

9. Exercises on the Application of the Axioms.

1. A man received for his day's wages \$2 and then spent \$2. How much money out of his day's wages did he have left?

2. A boy who had 10 marbles lost 6; then he bought 6. How many did he have then?

3. $y - 6 + 6 = ?$ $x + 8 - 8 = ?$ $z - a + a = ?$

4. If $5x = 15$, what does $5x + 7$ equal? What axiom applies?

5. If $x - 3 = 10$, how can we find the value of x? What axiom applies? What does x equal?

6. Suppose $x = 4$, how can we change the equation so that the right side shall be 5?

7. If $x + 8 = 17$, what does $x + 12$ equal? $x + 4$? What does x equal?

8. If $x = 3$, what does $8x$ equal? What axiom?

9. If $x = 5$, what does $4x + 7$ equal?

10. If $\frac{x}{3} = 7$, what does x equal? What axiom?

11. If $\frac{4x}{5} = 20$, what does $4x$ equal?

12. If $5x = 30$, what does x equal? What axiom?

13. Find $\frac{x}{2}$ if $\frac{3x}{2} = 15$. Find x.

14. If $\frac{3x}{5} - 8 = 10$, how can we get the value of $\frac{3x}{5}$? What axiom? How then can we get the value of $3x$? How then the value of x?

15. How from $\frac{3x}{5} = 18$ can we get the value of $\frac{x}{5}$? How then the value of x?

10 Express in the form of equations the following statements.

1. c is equal to a added to b.
2. Thirteen exceeds seven by six.
3. The excess of 9 over 5 is equal to 4.

4. The excess of a over b is equal to c.
5. c is 3 less than $a + b$.
6. a is as much greater than 5 as 30 is greater than c.
7. a is four times as great as b.
8. Three times a exceeds 29 by b.
9. Four times y is equal to the excess of x over 2.
10. a subtracted from a is equal to zero.

11. Exercises in the Use of Algebraic Language.

1. If a pear cost x cents and an orange $2x$ cents, what will represent the cost of both? Ans. x cents $+ 2x$ cents.

2. If John has marbles to the number of $3x$, and Henry to the number of $5x$, how many marbles do they have together?

3. If \$2, \$3, and \$4, are together \$9, what are $2x$, $3x$, and $4x$ together?

4. If A earns \$$2x$, B \$$3x$, C \$$4x$, in a day, how many dollars do A, B, and C together earn in a day?

5. If x represent A's money in dollars, and B has twice as much as A, and C three times as much as B, what will represent the entire amount of their money?

6. If an apple cost x cents, what will represent the cost of 2 apples? Of 3 apples? Of 4? Of 5?

7. If A can travel y miles in an hour, how many miles can he travel in 2 hours? In 5 hours? In 7? In 9?

8. If a contractor pays x dollars a day to each of his men, what does he pay to each man for 6 days' work? What does he pay 6 men for a day's work? To 3 men for 2 days' work? To 2 men for 3 days' work?

9. What will 12 yards of cloth cost at \$$2x$ a yard? 8 yards at \$$3x$ a yard? 3 yards at \$$8x$ a yard?

10. If 3 bags of coffee cost $3x$ dollars, what does one bag of coffee cost? 5 bags?

11. If $8x$ dollars is the price of 4 bushels of corn, what is that a bushel?

12. What must I pay for 10 barrels of flour if I can buy 8 barrels for $24x$ dollars?

13. Mr. Wallace, who is $8x$ years old, has a son Arthur half his age. Arthur has a daughter Alice also half of his age. How old is Alice?

14. If 8 bushels of corn are worth $16x$ dollars, and 5 barrels of apples $15x$ dollars, how many bushels of corn ought to be given for 6 barrels of apples?

12. PROBLEMS.

1. Two boys, James and Henry, had together 50 cents, and Henry had 4 times as many as James, how many did each have?

If now we knew how many James had, we could find the number Henry had. Suppose we represent this now unknown number by x. Then x will represent the number of cents James had, and 4 times x, or $4x$, the number of cents Henry had, and $x + 4x$ the number of cents both together had.

Then $x + 4x = 50$

or $5x = 50$

and $x = 10$, the no. of cents James had.

$4x = 40$, " " Henry "

2. Three men, A, B, and C, form a company with a capital of \$3000. C put in three times as much as A, and B twice as much as A. How many dollars did each put in?

Let $x =$ the no. of dollars A put in,

then $2x =$ " " B "

and $3x =$ " " C "

and $x + 2x + 3x =$ " " they all put in.

But they all put in \$3000;

$$\therefore \quad x + 2x + 3x = 3000$$
$$6x = 3000$$
$$x = 500$$
$$2x = 1000$$
$$3x = 1500$$

Therefore, A put in \$500, B \$1000, and C \$ 1500.

3. The sum of two numbers is 84, and the greater is twice the less. What are the numbers? Ans. 28 and 56.

4. Two men, A and B, contribute to a fund \$40, of which B gives \$10 more than A. What does each give?

Let $x =$ the no. of dollars A gives,
then $x + 10 =$ " " B "
and $x + x + 10 =$ " " A and B give.
But A and B together give \$40.

$\therefore$ $x + x + 10 = 40$
or $2x + 10 = 40$
Now $10 = 10$
By subtraction $2x = 30$ (Ax. 2.)
$x = 15$, the no. of dollars A gives,
and $x + 10 = 25$, " " B "

5. The number of years in the ages of two boys, Dick and Jack, added together make 31, and Dick is five years older than Jack. What are the ages of Dick and Jack?

Let $x =$ the no. of years in Dick's age,
then $x - 5 =$ " " Jack's "
and $x + x - 5 = 31$
or $2x - 5 = 31$
But $5 = 5$
By addition $2x = 36$ (Ax. 1.)
$\therefore$ $x = 18$, the no. of years in Dick's age,
and $x - 5 = 13$, " " Jack's "

6. In a school there are 165 pupils and twice as many boys as girls. How many boys are there?

7. Four times a certain number added to three times the same number gives 56. Find the number.

8. A farmer being asked how many sheep he had, said that if he had 5 times as many more he should have 240. How many had he?

9. The sum of two numbers is 344, and the greater is 7 times the less. What are the two numbers?

10. A horse and carriage are together worth \$320, and the horse is worth \$100 more than the carriage. What is each worth?

11. A farmer has a horse, a cow, and a sheep. The horse is worth 3 times as much as the cow, and the cow 8 times as much as the sheep, and all together are worth \$198. How much is each worth? Ans. Sheep, \$6; cow, \$48; horse, \$144.

The answers to examples should be *verified*, that is, shown to fulfil the given conditions. Thus, in Prob. 11, it is stated, (1) That the horse is worth 3 times as much as the cow, (2) That the cow is worth 8 times as much as the sheep, and (3) That all together are worth \$198. These conditions are all fulfilled by the prices \$144, \$48, and \$6, respectively.

12. Two partners, A and B, have in the firm property to the value of \$6546, and B has \$500 more than twice as much as A. What is the share of each?

13. The sum of the number of years in the ages of a father and his son is 70, and 2 years ago the father's age was twice the son's. What is the age of each? Ans. Son, 24; father, 46.

14. Three orchards bore 48 barrels of apples, of which the first bore twice as many as the second, and the third as many as the other two. How many barrels did each bear?

15. Three men, A, B, and C, received \$138 for digging a ditch. A dug 3 rods while B was digging 2 and C 1. How much should each receive?

16. A man is three times as old as his son, and his son twice as old as his daughter, and the sum of the number of years of their ages is 50. What is the age of each?

17. A, B, and C hired a house for \$500, of which A was to pay twice as much as B, and B \$60 less than C. How much was each to pay?

18. Two flocks of sheep are equal in number, but if 45 are transferred from one to the other, one will have four times as many as the other. How many were there in the original flocks?

CHAPTER II.

13. REDUCTION OF SIMPLE EQUATIONS.

Find the value of x in the following equations:

1. $5x - 9 = 3x + 3$ (1)

If we add 9 to both sides of (1) we have,

$$5x - 9 + 9 = 3x + 3 + 9 \qquad (2)$$

or

$$5x = 3x + 3 + 9 \qquad (3)$$

If we subtract $3x$ from both sides of (3) we have,

$$5x - 3x = 3x - 3x + 3 + 9 \qquad (4)$$

or

$$5x - 3x = 3 + 9 \qquad (5)$$

$\therefore$

$$2x = 12$$
$$x = 6$$

2. $6x + 3 = 10x - 1.$ Ans. $x = 1.$

3. $8x - 15 = 6x - 3.$ Ans. $x = 6.$

4. $9x + 8 = 10x + 6.$ 2

5. $7 - 4x = 7x - 4.$ /

6. $x + 3x - 5 = 7.$

7. $5x - 4 - 6 = 3x.$

14. The parts of an algebraic expression connected by the signs + or − are called *terms.* Thus, in Ex. 7, there are four terms, $5x$, -4, -6, and $3x$.

15. The equation given in Ex. 1,

$$5x - 9 = 3x + 3 \qquad (1)$$

it will be seen becomes in equation (5)

$$5x - 3x = 3 + 9 \qquad (5)$$

That is the terms -9 and $3x$ appear in equation (5), each on the side opposite to that on which it appears in equation (1), and with an opposite sign. For convenience this change of side and sign is called *transposition.* According to this any term can be erased from one side, provided we put it on the other side with the opposite sign.

8. $\frac{x}{2} + 4 = 16.$ (1)

If we subtract 4 from both sides of (1), we have,

$$\frac{x}{2} = 12 \qquad (2) \qquad \text{(Ax. 2)}$$

If we multiply both sides of (2) by 2, we have,

$$x = 24 \qquad \text{(Ax. 4)}$$

16. This last process is called *clearing the equation of fractions.*

9. $\frac{x}{3} + 5 = \frac{x}{9} + 7.$ (1)

If we multiply (1) by 9, we have,

$$3x + 45 = x + 63$$
$$\therefore \quad 2x = 18$$
$$x = 9$$

10. $\frac{x}{2} - 6 = \frac{x}{4} - 2.$

11. $\frac{x}{5} + 8 = 15.$

12. $\frac{2x}{3} - \frac{x}{6} = 6.$

13. $\frac{4x}{7} + 5 = \frac{3x}{7} + 8$ (1)

$$\frac{4x}{7} - \frac{3x}{7} = 8 - 5 \qquad (2)$$
$$\frac{x}{7} = 3 \qquad (3)$$
$$x = 21$$

NOTE. It is often better as in Ex. 13 above to unite terms before clearing of fractions.

14. $\frac{3x}{5} + 8 = \frac{x}{5} + 12.$

15. $\frac{5x}{3} - \frac{2}{3} = \frac{2x}{3} + \frac{1}{3}.$

16. $\frac{x}{3} - 1 = \frac{x}{6} + 2.$

17. $\frac{x}{4} + \frac{x}{2} - \frac{x}{5} = 11.$

18. $\frac{3x}{5} - 10 = \frac{7x}{10} - 13.$

19. $2x - 8\frac{1}{4} = \frac{35}{4} - \frac{x}{8}.$

20. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 26.$

21. $\frac{x}{2} - \frac{x}{6} = 12 - \frac{x}{3}.$

22. $9\frac{1}{4} + \frac{x}{2} - \frac{2x}{7} = 6x + 7\frac{1}{4} - \frac{11x}{2}.$

23. $x - \frac{x}{5} + \frac{x}{3} = \frac{x}{7} + \frac{5x}{6} + 33$ (1)

Multiply (1) by 6.

$$6x - \frac{6x}{5} + 2x = \frac{6x}{7} + 5x + 198 \quad (2)$$

Transpose and unite.

$$3x - \frac{6x}{5} - \frac{6x}{7} = 198 \quad (3)$$

Multiply (3) by 7.

$$21x - \frac{42x}{5} - 6x = 1386$$

or $$15x - \frac{42x}{5} = 1386 \quad (4)$$

Multiply (4) by 5.

$$75x - 42x = 6930$$

or $$33x = 6930$$

$\therefore$ $$x = 210$$

NOTE. We might have multiplied (1) by $6 \times 7 \times 5$, or 210, at once.

24. $\frac{x}{5} - \frac{x}{3} + 21\frac{1}{4} = \frac{3x}{8} - 39\frac{3}{4}$.

17. Numbers are often grouped by different forms of the Bracket, ( ), [ ], { }, and the Vinculum, ——. Thus, $(a + b - c)$, $[a + b - c]$, $\{a + b - c\}$, $\overline{a + b - c}$, indicate that a, b, and $-c$, are to be considered as one whole, and subjected to the same operation.

The expression $16 + (8 - 3)$ means that $8 - 3$, or 5, is to be added to 16, or $16 + 5 = 21$. The result is the same if we first add 8 to 16 and from the sum subtract 3. 16 and 8 are 24; 24 minus 3 are 21.

That is, $16 + (8 - 3) = 16 + 8 - 3 = 21$

The expression $16 - (8 + 3)$ means that the sum of 8 and 3 is to be subtracted from 16, or $16 - 11 = 5$. In this case we can subtract 8 from 16, which leaves 8, and then from 8 subtract 3 more, which, as before, leaves 5.

That is, $16 - (8 + 3) = 16 - 8 - 3 = 5$

The expression $16 - (8 - 3)$ means that $8 - 3$, or 5, is to be subtracted from 16, or $16 - 5 = 11$. If now we first subtract 8 from 16 the remainder is 8; but we are not to subtract 8 but 3 less than 8. We have then subtracted 3 too many, and therefore the remainder 8 must be 3 too small, and the true remainder must be $8 + 3 = 11$.

That is, $$16 - (8 - 3) = 16 - 8 + 3 = 11$$

The principle is evidently the same whatever the particular numbers may be. It appears then that if a parenthesis is removed with the sign + before it, the signs of all the terms remain unchanged; but if the sign − is before the parenthesis, the signs of each term within the parenthesis is changed from + to −, or − to +.

NOTE. The first term in a parenthesis having no sign before it is of course +.

18. When an expression of several terms is to be multiplied or divided, each term must be multiplied or divided. Thus, 5 times $3 + 7$ is $15 + 35 = 50$. For $3 + 7 = 10$, and 5 times 10 is 50. $35 - 14$ divided by 7 is $5 - 2$, or 3. For $35 - 14 = 21$, and 21 divided by 7 is 3.

1. $4(5 + 8) = 20 + 32 = 52 = 4 \cdot 13$.
2. $9(8 - 3) = 72 - 27 = 45 = 9 \cdot 5$.
3. $(18 + 12) \div 6 = 3 + 2 = 5 = 30 \div 6$.
4. $(42 - 18) \div 3 = 14 - 6 = 8 = 24 \div 3$.

The line between the numerator and denominator of a fraction acts also as a vinculum.

Thus, $$27 - \frac{16 - 4}{4} = 27 - (4 - 1) = 27 - 4 + 1 = 24$$

5. $38 + \dfrac{18 - 6}{3} = 38 + (6 - 2) = 42$.

6. $25 - \dfrac{25 - 15}{5} = ?$ 7. $76 - \dfrac{33 + 6}{3} = ?$

19. Find the value of x in the following equations:

1. $\dfrac{x+5}{3}+\dfrac{x}{5}=25-\dfrac{x+26}{2}$. (1)

Multiplying (1) by 6,

$$2x+10+\frac{6x}{5}=150-3x-78$$

Transposing and uniting,

$$5x+\frac{6x}{5}=62$$

$$25x+6x=31x=310$$

$$x=10$$

2. $4=\left(\dfrac{x}{3}+7\right)-5$.

3. $x-\dfrac{2x+1}{5}=2+\dfrac{x+3}{3}$.

4. $x-\left(\dfrac{x}{2}-\dfrac{x}{3}\right)=5$.

$$x-\frac{x}{6}=\frac{5x}{6}=5$$

$$x=6$$

5. $3x-\dfrac{9x-29}{4}=27-\dfrac{6x+11}{5}$.

6. $3x=\dfrac{117-x}{4}-\dfrac{x-95}{3}$.

7. $5-\dfrac{x+6}{5}=4x-\dfrac{11+11x}{3}$.

8. $3x-\dfrac{8-x}{3}-7=\dfrac{3(x+4)}{18}-4(x-1)$.

9. $\dfrac{5-x}{2}-\dfrac{4x+5}{3}=\dfrac{19+5x}{4}-7$.

10. $\dfrac{7}{6}-\dfrac{9}{5x+1}=\dfrac{2x+5}{6}-\dfrac{4x-1}{12}$.

NOTE. Multiply by 12; then transpose and unite.

11. $\dfrac{x+16}{6}-\dfrac{x+4}{6}=21-\dfrac{x-1}{2}$.

Uniting the fractions in the first member,

$$\frac{12}{6}=2=21-\frac{x}{2}+\frac{1}{2}$$

$$\frac{x}{2}=\frac{39}{2}$$

$$x=39$$

12. $\frac{x-6}{4}+7-\frac{2x-3}{3}=-6.$

13. $\frac{22-x}{5}+\frac{x-1}{6}=6-\frac{3+x}{5}.$

NOTE. Transpose $\frac{22-x}{5}$ and unite it with $-\frac{3+x}{5}$.

14. $19-\frac{284-4x}{3}=\frac{3x-75}{6}-\frac{2x-22}{12}.$

NOTE. First write the equation thus:

$$19-\frac{284-4x}{3}=\frac{x-25}{2}-\frac{x-11}{6}$$

15. $x-\frac{7x+5}{7}=\frac{6x-30}{7x-7}-1.$

NOTE. First multiply by 7; then transpose and unite.

16. $4x-\frac{20x+21}{4}+\frac{1}{4}=\frac{x-12}{3}-5.$

NOTE. First write the equation thus:

$$4x-5x-5\tfrac{1}{4}+\tfrac{1}{4}=\frac{x}{3}-4-5$$

Then
$$-x=\frac{x}{3}-4$$
$$4=\frac{4x}{3}$$
$$3=x$$

17. $\frac{8x-13}{5}-\frac{13-3x}{10}=6x-7-\frac{3x+2}{5}.$

18. $\frac{x+1}{6}-\frac{4x+5}{5}=1-\frac{5x-5}{4}.$

19. $7-\frac{3x+5}{2}-\frac{3(x+5)}{4}=5-\frac{5x+3}{2}.$

20. $8-\frac{5x}{3}-\frac{5x+3}{6}=4x-15+\frac{3x+3}{4}.$

21. $\frac{2x-38}{3}+\frac{23-x}{4}+7=2x-11-\frac{3x-19}{2}.$

22. $\frac{x+8}{6}-\frac{5x-6}{7}+\frac{4x-6}{5}=4-\frac{22-4x}{3}.$

CHAPTER III.

20. ORAL EXERCISES.

1. How old shall I be in a years if I am 25 years old now?

2. How old was I 6 years ago if I am x years old now?

3. How old shall I be in x years if I am y years old now?

4. If John has a cents and 20 cents, how many cents has he?

5. If I am $2y$ years old, and in 4 years shall be twice as old as my son, how old is my son?

6. What number exceeds a by 4?

7. Find a number less than x by 5.

8. If a is a number, how is the number that is one half of a to be expressed? Three times as great?

9. What number exceeds n by m?

10. If x is one part of 12, what is the other part?

11. If of two factors whose product is 24, x is one, what is the other?

12. If q is the quotient, and 8 is the divisor, what is the dividend?

13. If d is the divisor, q is the quotient, and r the remainder, what is the dividend?

14. What number is half as great again as y?

15. If 35 contains x five times, what is the value of x?

16. If y apples cost 25 cents, how many apples can be bought for $1?

17. If d is divided into two parts, and 8 is one of them, what is the other?

18. If $6x$ is the product of two numbers, and 6 is one of them, what is the other?

19. If I walk x miles in 5 hours, how many miles do I walk in one hour?

20. If I walk x miles in 3 hours, how long does it take me to walk 1 mile?

21. If I can row 8 miles in h hours, what is my rate? How many hours does it take me to row one mile? How many miles can I row in one hour?

22. If I can row m miles in h hours, what is my rate?

23. If x men can build a wall in 9 days, how long will it take one man?

24. If 9 men can dig a ditch in x days, how long will it take one man?

25. If x men can dig a ditch in y days, how long will it take one man?

26. If Mr. Smith is $a + b$ years old to-day, how old was he a years ago? How old b years ago?

27. If a man was y years old ten years ago, how old will he be z years hence?

21. WRITTEN EXERCISES.

Should any pupil find difficulty in answering the questions, let him substitute figures for the letters.

1. A boy had x cents, and then spent 5, and afterwards doubled his money. How many cents did he then have?

2. A farmer having $2x$ sheep buys 24 more, and then sells half of them. How many sheep has he left?

3. John is four years older than Henry, and Henry five years older than James. If x years is Henry's age, what represents the ages of John and James, respectively.

4. A has x dollars, and B has \$25 more than twice as much. B gives A \$20, and then each doubles the money he has. How much has each? How much both?

5. Two boys, Robert and George, had together \$20. At the end of a year Robert has gained \$4 plus what he had at first,

and George has gained $25 less than three times what he had at first. If x represent the number of dollars Robert had at first, what will represent the number each has at the end of the year?

6. A has $4x$ dollars, and B half as much; then A lends B $10; then both A and B double what they have, and B repays A. Find what each of them had.

7. If a merchant begins trade with x dollars, and doubles his stock every year, lacking $500, what will he have at the end of the third year?

8. Monday A has y dollars. Every night, including Monday night, he gives his wife $2, and each day for the next 4 days doubles what he has left. How much will he have Saturday morning?

9. If n is any integral number, $2n$ is always an even number. What is the next even number after $2n$? The next before it?

10. What is the next odd number after $2n$? The second odd number? The second odd number before $2n$?

11. Find the sum of three consecutive odd numbers of which the middle one is $2n - 1$.

12. A is x years old, C's age is three times B's, and B's is twice A's. Find C's age.

13. If a man was m years old x years ago, how old will he be n years hence?

14. A boy is n years old, and eight years hence he will be half the age of his father. How old is the father now?

15. How old is Mr. Child, if y years ago he was t times as old as his son, who was then x years old?

16. What number exceeds the sum of a and b by their difference, if a is greater than b?

17. If a horse eats o bushels of oats and c bushels of corn a month, how many bushels of oats and corn will he eat in m months?

18. If a train runs m miles in h hours, what is its rate? Express it in two ways. (See Ex. 21, § 20.)

19. If a post $8x$ feet in length is half in the water, and one eighth in the mud, and the rest above the water, how many feet are above the water?

20. How far will a person walk in 50 minutes, if he walks m miles in h hours?

21. How long will it take a person to walk a miles, if he walks 25 miles in b hours?

22. How long will it take x men to cut y acres of grass, if each man cuts z acres a day?

23. A and B start from the same point and walk in exactly opposite directions at the rate of a and b miles an hour, respectively. How far apart will they be at the end of one hour? 5 hours? 15 minutes?

24. If A and B start from the same point and walk in the same direction at rates, respectively, of a and b miles an hour, how far apart will they be at the end of one hour? At the end of 2 hours and 20 minutes?

25 If A can do a piece of work in d days, what part of the work can he do in one day? In two days? In b days?

26. If a man can do a piece of work in $\frac{x}{y}$ days, what part of the work can he do in one day?

27. A can do a piece of work in a days, and B in b days. What part of the work can they do together in one day? How long will it take them to do the whole work?

28. If a pipe discharges x gallons of water in y hours, how many does it discharge in one hour? How long does it take to discharge one gallon?

29. A can do a piece of work in a days, B in b days, and C in c days. What part of the work can they do together in one day? How long will it take them, working together, to do the work?

PROBLEMS

PRODUCING SIMPLE EQUATIONS CONTAINING BUT ONE UNKNOWN NUMBER.

22. The Solution of a Problem in Algebra consists:

1st. In reducing the statement to the form of an equation;

2d. In reducing the equation so as to find the value of the unknown numbers.

1. A father and son have property worth $66, and the father's part is double the son's. What is the part of each?
Ans. Son's part, $22. Father's part, $44.

2. The sum of three numbers is 49. The greatest is four times the least, and the intermediate number is twice the least. What are the numbers? Ans. 7, 14, 28.

3. Two boys have $24. If the first has double the second, how many dollars has each?
Ans. The first, $16; the second, $8.

4. A father's age is three times that of his son, and their ages added together amount to 48 years. How old is the son?

5. A carriage-horse cost three times as much as the carriage, and the price of the two was $360. Find what each cost.

6. In a school of 84 pupils, twice as many study geography as grammar, and twice as many study arithmetic as geography. How many pupils are there in the respective classes?
Ans. 12, 24, and 48.

7. The distance between two places, A and B, is four times the distance between B and C. If the difference of these distances is 51 miles, find the number of miles from A to B.

8. From two towns, 77 miles apart, two carriers at the same time start toward each other at the rate of 5 and 6 miles an hour, respectively. In how many hours will they meet? Dick

9. At the end of the fourth day the captain of a vessel, sailing from Boston to London, found that he had sailed 240 miles. The second day he sailed twice as far as the first, and the third as far as the first and second days together, but the fourth, by a severe storm, he was driven back as many miles as he sailed the second day. How many miles did he sail the first day ?

Ans. 60.

10. A drover sold 7 sheep and 5 lambs for $57, receiving twice as much for a sheep as for a lamb. How much did he receive for each ?

11. Three men hire a pasture for the summer for $81. The first puts in two cows; the second, three; and the third, four. If the cows remain in the pasture the same length of time, how much ought each to pay ?

12. Three men, A, B, and C, built 57 rods of fence. A built 5 rods a day; B, 3; and C, 2. C worked twice as many days as B, and B twice as many as A. How many days did each work ? Ans. A, 3 days; B, 6 days; C, 12 days.

13. The difference between two numbers is 7, and their sum is 23. What are the numbers. Ans. 8 and 15.

14. To four times a certain number I add 12, and obtain 32. What is the number ?

15. Two persons, A and B, divide $50 between them, so that A has $6 more than B. What are their shares ? 22, 28

16. Three checks are together worth $80. The first is worth $25 less than the second, and $20 more than the third. What are their values ? Ans. $25, $50, $5.

17. A father left his three sons $19000, the eldest to have $2000 more than the second, and $3000 more than the third. Find the share of each.

18. In a hamlet containing 90 persons, there are 4 less men than women, and 10 more children than adults. How many men, women, and children are there ?

19. A has two dollars less than three times what B has, and they both together have \$62. What has each?

20. Divide \$300 among A, B, and C, so that A shall have twice as much as B, and B, \$20 more than C.

Ans. A, \$160; B, \$80; C, \$60.

21. A man walks 8 miles, then goes a certain distance by coach, and then three times as far by train as by coach. If the whole journey is 88 miles, how far did he travel by train?

22. Find a number such that, if 10 is taken from its double, and 20 from the double of the remainder, there will be 40 left.

Ans. 20.

23. A father said to his son, "3 years ago I was three times as old as you, but in 9 years I shall be exactly twice as old." What were their ages?

Let $x =$ no. of years in the son's age;

then $(x-3)\,3+3 =$ " " father's age.

$x+9 =$ " " son's age 9 years hence.

$(x-3)\,3+12 =$ " " father's age 9 years hence.

$$(x-3)\,3+12 = 2\,(x+9)$$

$$3\,x+3 = 2\,x+18$$

$x = 15$ no. of years in the son's age.

$(x-3)\,3+3 = 39$ " " father's age.

24. A certain number of two figures whose sum is 8 will have the order of the figures reversed if 18 is added to it. What is the number?

Ans. 35.

Let $x =$ the tens' figure;

then $8 - x =$ the units' figure.

$10\,x + 8 - x = 9\,x + 8 =$ the number;

and when the order of the figures is reversed,

$10\,(8-x) + x = 80 - 9\,x =$ the number.

$$9\,x + 8 + 18 = 80 - 9\,x$$

$$18\,x = 54$$

$x = 3$ the tens' figure.

$8 - x = 5$ the units' figure.

25. A train carries 185 travellers from Liverpool to London. The number of travellers of the third class is 19 more than the number of the other two together, and of the second class, 13 more than of the first class. Find the number of each class.

Ans. 35; 48; 102.

26. A pupil who said he had received the first prize in Arithmetic was asked how many other prizes he had received. He replied: "If from 4 times the number of my other prizes, you subtract 3, then add 5 more than double the number, the result will be 14." Find the number of all his prizes. Ans. 3.

27. "What is the date of your birth?" said a pupil to his master. The master answered: "The sum of the four figures is 18; the tens' figure is 2 less than the hundreds' and 3 more than the units'." What was the date? Ans. 1863.

NOTE. The thousands' figure is of course known.

28. A person gave $8 to 4 families. To the second, $0.80 less than twice what he gave to the first; to the third, $1.40 less than three times what he gave to the first; and to the fourth, $0.60 more than twice what he gave to the first. How much did he give to each? Ans. $1.20; $1.60; $2.20; $3.

29. A student bought four books. The price of the second was $0.60 less than twice the price of the first; of the third, $0.20 more than twice the price of the first; of the fourth, $0.80 less than three times the price of the first; and the sum paid for the first three was $2.80 more than the price of the fourth. Find the price of each book. Ans. $1.20; $1.80; $2.60; $2.80.

30. A boy started from home to walk to Boston. The second day he walked 10 miles less than twice the distance he walked the first day; the third day, three halves as much as the first day; the fourth day, 16 miles less than three times the distance he walked the first day, and reaches the city. Moreover the distance walked on the second and third days together was a mile less than on the first and last days together. How far was his home from Boston? Ans. 79 miles.

31. Divide $760 among A, B, and C, so that B may have $50 more than A and $135 less than C.

32. A man paid $22.20 to two boys who had worked the first, 12 days, and the second, 15. The second received $0.40 a day more than the first. Find the daily wages of each.

Ans. $0.60; $1.

33. A man paid $40 with 67 pieces of silver, dollars and quarters. How many of each did he use?

Ans. 31 dollars, and 36 quarters.

34. Two persons have the same annual salary. The first saves $\frac{1}{5}$ of his, and the second spends $160 more than the first annually, and at the end of 3 years is in debt $170.40. What is this salary? Ans. $516.

35. A sum of money is given to A and B in such a way that A has $\frac{3}{4}$ as much as B, and $\frac{1}{10}$ of A's money plus $\frac{8}{10}$ of B's is $21. Find the share of each?

36. Two casks contain one, 90 gallons, and the other, 110 gallons of oil. The same quantity is drawn from each, and then the first contains $\frac{3}{4}$ as much as the second. What is the quantity drawn out?

37. A journeyman agreed to work during the month of September (four Sundays excepted) for $3.50 a day; but for every day he was idle he was to forfeit $1.50. He received $61. How many days was he idle? Ans. 6.

38. How many days could the journeyman named in Example 37 have worked and yet have been entitled to no pay?

Ans. $7\frac{1}{5}$.

39. Two towns, A and B, are 100 miles apart. At A, coal costs $5 a ton, and at B, $5.25 a ton. If the freight is a cent and a quarter a ton a mile, at what point between A and B does coal cost the same? How does this cost compare with the cost at other points between A and B?

Ans. 1st. 60 miles from A; 2d. Dearest.

40. In a certain factory \$4560 are paid each week as wages. The workmen are divided into three classes. The first class receive \$6 each a week; the second, \$7; and the third, \$8. There are 4 of the first class for 12 of the second, and 4 of the second for 5 of the third. Find the number of workmen in each class. Ans. 1st class, 80; 2d, 240; 3d, 300.

41. A father 43 years old has two sons, 10 years and 6 years old, respectively. How long ago was the father 4 times as old as the sum of the ages of his two sons?

42. A man engaged a coachman for \$300 a year and his livery. At the end of 8 months the coachman left, receiving \$188 and keeping the livery. What was the cost of the livery? Ans. \$36.

43. There are two pieces of land whose combined areas are $33\frac{3}{8}$ acres, and $\frac{7}{8}$ of one is equal to $\frac{10}{11}$ of the other. Find the area of each? Ans. 18 acres, and $15\frac{3}{8}$ acres.

44. Two casks contain together 55 gallons of kerosene. If a third part of the first and a fifth part of the second are drawn out, the casks will contain an equal amount. How many gallons are there in each?

45. Two boys save, one a third, and the other a fourth of his income. The sum of their gain is \$80 and the sum of their income is \$270. What is the income of each? Ans. \$150; \$120.

46. Two persons, A and B, receive by will, together, \$3660. A spends $\frac{2}{5}$ of his, and B $\frac{3}{7}$ of his, and then A has twice as much as B. How much did each receive? Ans. A, \$2400; B, \$1260.

47. A dealer in grain wished to fill a bin that holds 217 bushels with a mixture of oats and barley. The oats cost \$0.45, and the barley \$0.52, a bushel. How many bushels of each must he take in order that a bushel of the mixture may cost \$0.50 a bushel? Ans. Oats, 62; barley, 155.

48. Divide \$1298 among 4 persons so that the first shall have \$20 more than the second; the second, \$48 more than the third; and the third, \$70 more than the fourth.

Ans. 1st, \$381; 2d, \$361; 3d, \$313; 4th, \$243.

49. Divide \$2000 among 4 persons so that the first may have \$400 less than twice what the second has; the second, \$600 less than three times as much as the third has; and the third, \$800 less than six times as much as the fourth has.

Ans. 1st, \$800; 2d, \$600; 3d, \$400; 4th, \$200.

50. Divide \$360 among 3 persons so that the second may have \$30 more than $\frac{2}{5}$ as much as the first, and the third, \$24 less than $\frac{3}{4}$ as much as the second.

Ans. 1st, \$195; 2d, \$108; 3d, \$57.

51. Divide \$1120 among 5 persons so that the second may have \$40 more than double the first; the third, \$80 less than three times the first; the fourth, \$30 more than half as much as the second and third together; and the fifth, \$95 more than a fourth as much as the four others together.

Ans. 1st, \$100; 2d, \$240; 3d, \$220; 4th, \$260; 5th, \$300.

52. A sum of money was divided among 4 persons so that the first had $\frac{1}{5}$ of the whole; the second, $\frac{4}{9}$ of the remainder; the third, $\frac{2}{5}$ of the second remainder; and the fourth, the rest, or \$480. What was the sum? Ans. \$1800.

53. Divide \$9 among 9 persons: 1 man, 3 women, and 5 children, in such a way that the man may have $\frac{5}{3}$ as much as a woman, and a woman $\frac{5}{2}$ as much as a child.

Ans. The man, \$2.25; a woman, \$1.35; a boy, \$0.54.

54. From Fitchburg to Boston by rail is 50 miles. If coal can be bought in Boston at \$6.60 a ton, and in Fitchburg at \$6.75, and it costs for freight a cent and a half a mile a ton, at what point on the line would coal cost exactly the same whether it came from Boston or Fitchburg?

Ans. 20 miles from Fitchburg.

55. A box contains \$15 in silver, dollars and quarters. If there are 21 pieces of money, how many of each kind are there?
Ans. 13 dollars; 8 quarters.

56. The toll on a certain bridge for a man on foot is 1 cent; for a man on horseback, 2 cents; for a carriage with one horse, 3 cents; and for a carriage with two horses, 4 cents. On a certain day there passed over the bridge carriages with two horses, $\frac{2}{5}$ as many as carriages with one horse; carriages with one horse, $\frac{3}{11}$ as many as men on horseback; men on horseback, $\frac{5}{27}$ as many as men on foot; and the receipts were \$57.12. How many of each variety were there?

Ans. 3564 men; 660 men on horseback; 180 carriages with one horse; 72 carriages with two horses.

57. A merchant increased his capital each year by a third of what he had at the beginning of the year, and at the end of each year took out for his expenses \$1000. At the beginning of the fourth year his capital is double the original capital. What was the original capital? Ans. \$11100.

58. A banker at the end of a year had increased his capital by $\frac{2}{7}$ of itself; during the second year he lost $\frac{1}{8}$ of what he had at the beginning of the year; during the third year he added an amount equal to $\frac{1}{12}$ of his original capital; and during the fourth year the gain was equal to the gain of the first three years together. He then had \$28560. What was his original capital? Ans. \$20160.

59. A father and his two sons were paid for building a wall in such a way that the father received as much an hour as both sons, and the elder son 5 cents an hour more than the younger. The father worked 6 hours and 30 minutes; the elder son, 5 hours and 30 minutes; and the younger, 4 hours and 30 minutes. The sum received was \$2.90. What were the hourly wages of each? Ans. Younger son, \$0.10; elder, \$0.15; father, \$0.25.

60. A father is 35 years old and his son 6. How long before the father will be three times the age of his son?

61. Three partners divide the profits of the year so that the first has \$20 more than half the profits; the second has a fifth of the remainder and \$80; and the third, the rest, or \$104. What was the amount of the profits? Ans. \$500.

62. A florist started for market with roses which he expected to sell at 5 cents apiece. On the way he lost 8 roses, but found that by selling the rest at 6 cents apiece he should have the same amount of money as expected when he started. How many roses did he have when he started?

63. A boy when asked his age said, "In 16 years I shall be three times as old as I was 2 years ago." How old was he?

64. A person spent each day half the money he had left the day before, plus half a dollar, and at the end of the third day had nothing left. How much money did he have at first?

65. A market-man sells to one person half of his eggs plus 15 eggs; to a second, $\frac{4}{15}$ of the remainder plus 10 eggs; to a third, $\frac{1}{4}$ of the rest plus 9, and has none left. How many did he have at first?

66. "How old are you?" asked a son of his father. "In a year," answered the father, "I shall be three times as old as you will be, and in 19 years twice as old as you will be." How old were the father and the son?

Ans. Son, 17 years; father, 53.

67. A man sold a horse for 25% more than he paid for it. What he paid and what he received were together \$675. What did he pay?

68. A man invested $\frac{1}{4}$ of a certain sum of money at $3\frac{1}{2}\%$, $\frac{1}{5}$ at 4%, and the rest at $4\frac{1}{2}\%$. The annual interest of the three parts amounted to \$166. What was the sum invested?

69. What number must be added to the terms of $\frac{5}{7}$ that it ($\frac{5}{7}$) may become equivalent to $\frac{28}{33}$?

70. What one number must be added to the two terms of $\frac{15}{17}$ that it may become equivalent to $\frac{89}{97}$?

Dist. **71.** Two men start, the one at 6 A. M. and the other at 8.30 A. M., and walk directly toward each other. The first goes at the rate of a mile in 24 minutes, and the second, a mile in 20 minutes. When they meet they have travelled equal distances. What is the distance between the points of starting?

72. A contractor pays daily to 55 workmen, men and boys, \$81.25; to each man, \$1.75, and to each boy, \$0.75. How many men and how many boys does he employ?

73. A father is 30 years older than his son, and in 4 years will be 4 times as old. Find the age of each.

74. \$33456 are put at interest in two parts, such that the interest of the first part for 3 years at 4% is double the interest of the second part for 7 years at 5%. Find the parts.

75. \$12200 are put at interest in two sums, one at $4\frac{1}{2}$%, and the other at $3\frac{1}{2}$%, so that the annual interest is \$489. What are the two parts? Ans. \$6200 at $4\frac{1}{2}$%; \$6000 at $3\frac{1}{2}$%.

76. \$9000 are put at interest in two parts, one at $5\frac{1}{2}$%, and the other at 4%, so as to give an annual revenue of \$400.50. What are the two parts? Ans. \$2700 at $5\frac{1}{2}$%; \$6300 at 4%.

77. A person puts $\frac{3}{4}$ of a certain sum of money at interest at 4%, and the rest at 5%, the interest for 72 days is \$84.83. What is the sum? (Count 360 days a year.) Ans. \$9980.

78. A person puts $\frac{2}{3}$ of a certain sum of money at 5%, and the rest at $4\frac{1}{2}$%, and receives at the end of a year for interest and principal \$3145. What was the principal? Ans. \$3000.

79. A certain principal at 8% gives \$232 more interest annually than another principal \$1800 less, at 6%. What are the two principals? Ans. At 8%, \$6200; at 6%, 4400.

80. A man who has \$24000 spends a part in the purchase of a house, the rest he puts at interest, one third at 4%, and two thirds at 5%, and thus has an income of \$784 annually. Find the price of the house, and the two sums at interest.

Ans. House, \$7200. At int. 4%, \$5600; 5%, 11200.

CHAPTER IV.

SIMPLE EQUATIONS CONTAINING TWO OR MORE UNKNOWN NUMBERS.

ELIMINATION.

23. Elimination is the method of deriving from the given equations a new equation, or equations, containing one (or more) less unknown number. The unknown number thus excluded is said to be *eliminated.*

24. Equations containing only two Unknown Numbers.

1. Solve $\begin{cases} 2x + 5y = 31. & (1) \\ 3x + 4y = 29. & (2) \end{cases}$

$$y = \frac{31 - 2x}{5} \quad (3) \qquad 3x + 4\left(\frac{31 - 2x}{5}\right) = 29 \quad (4)$$

$$15x + 124 - 8x = 145 \quad (5)$$

$$y = \frac{31 - 6}{5} = 5 \quad (7) \qquad x = 3 \quad (6)$$

Transposing $2x$ in (1), and dividing by 5, we have (3), which gives an expression for the value of y. Substituting this value of y in (2), we have (4), which contains but one unknown number; that is, y has been eliminated. Reducing (4) we obtain (6), or $x = 3$. Substituting this value of x in (3), we obtain (7), or $y = 5$. Hence the following

Rule.

Find an expression for the value of one of the unknown numbers in one of the equations, and substitute this value for the same unknown number in the other equation.

This method of elimination is called *substitution.*

NOTE 1. After eliminating, the resulting equation is reduced as shown in Arts. 13–19. The value of the unknown number thus found must be substituted in one of the equations containing the two unknown numbers, and this reduced as shown in Arts. 13–19.

Solve the following equations by substitution.

2. $\begin{cases} 5x - 4y = -2. \\ 4x - 6y = -10. \end{cases}$

3. $\begin{cases} 6x - y = 11. \\ 5x + 3y = 36. \end{cases}$

4. $\begin{cases} 6x - y = 0. \\ 3x + 2y = 15. \end{cases}$

5. $\begin{cases} 7x + 2y = 24. \\ 6x - y = 7. \end{cases}$

6. $\begin{cases} 8x + 13 = 5y. \\ 3y - 4x = 11. \end{cases}$

7. $\begin{cases} 3x - y = 20. \\ 5x + 2y = 81. \end{cases}$

NOTE 2. The pupil should verify his results. It is not sufficient to substitute the results in one equation. Both answers may be wrong and yet verify in one of the given equations.

8. $\begin{cases} 2x + 5y = 23. \\ 4x + 3y = 25. \end{cases}$

9. $\begin{cases} 5x + 4y = 53. \\ 2x + 3y = 31. \end{cases}$

10. $\begin{cases} 4x - y = 3. \\ 3x + 2y = 5. \end{cases}$

11. $\begin{cases} 7x - 4y = 20. \\ 3x + 2y = 42. \end{cases}$

12. $\begin{cases} 13x - 3y = 6. \\ 8x + 5y = 79. \end{cases}$

13. $\begin{cases} 4y + 7x = 87. \\ 3x + 2y = 41. \end{cases}$

14. Solve $\begin{cases} 4x + 5y = 35. & (1) \\ 3x + 2y = 21. & (2) \end{cases}$

$$
\begin{array}{rlcrl}
12x + 15y = 105 & (3) & & & \\
\underline{12x + 8y = 84} & (4) & & & \\
7y = 21 & (5) & & 3x + 6 = 21 & (7) \\
y = 3 & (6) & & x = 5 & (8)
\end{array}
$$

If we multiply (1) by 3 and (2) by 4, we have (3) and (4), in which the coefficients of x are equal; subtracting (4) from (3), we have (5), which contains but one unknown number. Reducing (5) we have (6), or $y = 3$; substituting this value of y in (2), we obtain (7), which reduced gives (8), or $x = 5$. Hence the following

Rule.

Multiply or divide the equations so that the coefficients of the unknown number to be eliminated shall become equal; then, if the signs of this number are alike in both, subtract one equation from the other; if unlike, add the two equations together.

This method of elimination is called *combination*.

Note. The least multiplier for each equation will be that which will make the coefficient of the unknown number to be eliminated the least common multiple of the two coefficients of this number in the given equations. It is always best to eliminate that unknown number whose coefficients can most easily be made equal.

Solve the following equations by combination:

15. $\begin{cases} 6x - 5y = 10. \\ 5x + 3y = 37. \end{cases}$

16. $\begin{cases} 3x + 2y = 20. \\ 7x - 2y = 0. \end{cases}$

17. $\begin{cases} 11x + 4y = 19. \\ 4x + 3y = 10. \end{cases}$

18. $\begin{cases} 8x - 3y = 23. \\ 5x + 4y = 32. \end{cases}$

19. $\begin{cases} \dfrac{x}{3} - \dfrac{y}{4} = 2. \\ \dfrac{x}{5} + \dfrac{y}{6} = 5. \end{cases}$

20. $\begin{cases} \frac{2}{5}x + \frac{3}{7}y = 5. \\ \frac{3}{2}x - \frac{2}{3}y = \frac{17}{6}. \end{cases}$

25. Solve the following equations:

Note. Which method of elimination should be used depends upon the relations of the coefficients to each other. That one which will eliminate the number desired with the least work is the better.

1. $\begin{cases} 3x + 4y = 34. \\ 6x + 3y = 33. \end{cases}$

2. $\begin{cases} 4x - 3y = 0. \\ 5x + 2y = 69. \end{cases}$

3. $\begin{cases} \dfrac{x}{3} + \dfrac{y}{7} = 5. \\ \dfrac{x}{4} + \dfrac{y}{9} = \dfrac{23}{6}. \end{cases}$

1 4. $\begin{cases} 4x + \dfrac{y}{4} = 52. \\ \dfrac{x}{4} + 4y = 67. \end{cases}$

2 5. $\begin{cases} \dfrac{x}{3} + \dfrac{y}{4} = 9. \\ 3x + 4y = 109. \end{cases}$

3 6. $\begin{cases} \dfrac{x}{7} + \dfrac{y}{2} = 7. \\ \dfrac{x}{2} - \dfrac{y}{5} = 5. \end{cases}$

7. $\begin{cases} \dfrac{y}{4} - \dfrac{x}{5} = 3. \\ \dfrac{y - x}{10} = 1. \end{cases}$

8. $\begin{cases} \dfrac{3x}{5} + \dfrac{5y}{3} = 59. \\ \dfrac{5x}{3} - \dfrac{3y}{5} = 7. \end{cases}$

9. $\begin{cases} \dfrac{x - y}{2} + x = 15. \\ \dfrac{x + 3y}{5} + x = 15. \end{cases}$

10. $\begin{cases} \dfrac{x + 1}{3} + 4y = 5. \\ 4x - 2y = 31. \end{cases}$

11. $\begin{cases} \dfrac{x+3y}{3} + \dfrac{23}{6} = \dfrac{x+2y+4}{2}. \\ 9 - \dfrac{x+\ y}{4} = 5. \end{cases}$

NOTE. In the first equation above the $\frac{3y}{3}$ on the left is balanced by the $\frac{2y}{2}$ on the right, and the equation can be written at once

$$\frac{x}{3} - \frac{x}{2} = 2 - 3\tfrac{5}{6}. \qquad \therefore \quad \frac{x}{6} = \frac{11}{6}, \qquad \text{or } x = 11.$$

12. $\begin{cases} \dfrac{x+y}{5} - \dfrac{x-y}{5} = 4. \\ \dfrac{x-y}{3} + \dfrac{x+y}{4} = \dfrac{13}{4}. \end{cases}$

13. $\begin{cases} \dfrac{x-2y}{4} + \dfrac{y}{5} = \dfrac{x+y}{2} - 27. \\ \dfrac{x}{11} + \dfrac{y}{10} = \dfrac{x}{4} - \dfrac{y}{4}. \end{cases}$

14. $\begin{cases} 2x - \dfrac{y-8}{3} = 2y - 2. \\ 3y - \dfrac{2x+3}{3} = 2x + 3. \end{cases}$

26. Simple Equations containing more than two Unknown Numbers.

1. $\begin{cases} x + \ y + \ z = 6. \\ 2x + 3y + 4z = 20. \\ 3x - 4y + 6z = 13. \end{cases}$

$$\begin{array}{lll}
x+y+z=6 \ (1) & 2x+3y+4z=20 \ (2) & 3x-4y+6z=13 \ (3) \\
 & \underline{2x+2y+2z=12 \ (4)} & \underline{3x+3y+3z=18 \ (5)} \\
 & y+2z=8 \ (6) & -7y+3z=-5 \ (7) \\
 & & \underline{7y+14z=56 \ (8)} \\
x+2+3=6 \ (13) & y+6=8 \ (11) & 17z=51 \ (9) \\
x=1 \ (14) & y=2 \ (12) & z=3 \ (10)
\end{array}$$

Multiplying equation (1) by 2 gives equation (4), which we subtract from (2), and obtain (6); multiplying (1) by 3 gives (5), and subtract-

ing (5) from (3) gives (7). We have now two equations, (6) and (7), containing but two unknown quantities. Multiplying (6) by 7, we obtain (8), and adding (7) to (8), we obtain (9), which reduced gives $z = 3$. Substituting this value of z in (6), and reducing, we obtain $y = 2$. Substituting these values of y and z in (1), and reducing, we obtain $x = 1$.

2. Solve $\begin{cases} 2x - 3y + 4z = 5. \\ 3x + 2y = 21. \\ y + 4z = 7. \end{cases}$

$$
\begin{array}{lll}
2x - 3y + 4z = 5 \quad (1) & 3x + 2y = 21 \quad (2) & y + 4z = 7 \quad (3) \\
\underline{y + 4z = 7} & & \\
2x - 4y = -2 \quad (4) & & \\
\underline{6x + 4y = 42 \quad (5)} & & \\
8x = 40 \quad (6) & 15 + 2y = 21 \quad (8) & 3 + 4z = 7 \quad (10) \\
x = 5 \quad (7) & y = 3 \quad (9) & z = 1 \quad (11)
\end{array}
$$

Equation (1) has the three unknown numbers, x, y, z, while (2) has only x and y, and (3) only y and z. We therefore combine either (2) or (3) with (1) so as to eliminate one of the three numbers. We select (3) because z has the same coefficient in both (1) and (3). Subtracting (3) from (1) we have (4), which has x and y, the same unknown numbers as are in (2). Adding twice (2) to (4) we have (6). Dividing (6) by 8 we have (7), or $x = 5$. Substituting $x = 5$ in (2) we obtain (8), which reduced gives $y = 3$. Substituting $y = 3$ in (3) we obtain (10), which reduced gives $z = 1$.

Hence, for solving equations containing any number of unknown numbers, we have the following

Rule.

From the given equations deduce equations one less in number, containing one less unknown number; and continue thus to eliminate one unknown number after another, until one equation is obtained containing but one unknown number. Reduce this last equation so as to find the value of this unknown number; then substitute this value in an equation containing this and but one other unknown number, and, reducing the resulting equation, find the value of this second

unknown number; substitute again these values in an equation containing no more than these two and one other unknown number, and reduce as before; and so continue, till the value of each unknown number is found.

NOTE. The process can often be very much abridged by the exercise of judgment in selecting the unknown number to be eliminated, the equations from which the other equations are to be deduced, the method of elimination which shall be used, and the simplest equations in which to substitute the values of the numbers which have been found.

Find the values of the unknown numbers in the following equations:

3. $\begin{cases} x - y + 2z = 7. \\ 3x + 2y - z = 8. \\ 4x - 3y + z = 3. \end{cases}$ Ans. $\begin{cases} x = 2. \\ y = 3. \\ z = 4. \end{cases}$

4. $\begin{cases} 3x + 2y = z + 12. \\ y + 3z = 33. \\ 5x - y = 9. \end{cases}$ Ans. $\begin{cases} x = 3. \\ y = 6. \\ z = 9. \end{cases}$

5. $\begin{cases} 2x + 3y = 16. \\ 3y + 4z = 36. \\ 4x + 5z = 38. \end{cases}$ Ans. $\begin{cases} x = 2. \\ y = 4. \\ z = 6. \end{cases}$

6. $\begin{cases} \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 56. \\ \frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 43. \\ \frac{x}{4} + \frac{y}{5} + \frac{z}{6} = 35. \end{cases}$ Ans. $\begin{cases} x = 12. \\ y = 60. \\ z = 120. \end{cases}$

7. $\begin{cases} x + y + z + w = 15. \\ y + z + w + u = 18. \\ x + z + w + u = 17. \\ x + y + w + u = 16. \\ x + y + z + u = 14. \end{cases}$ Ans. $\begin{cases} x = 2. \\ y = 3. \\ z = 4. \\ u = 5. \\ w = 6. \end{cases}$

NOTE. If these equations are added together and the sum divided by 4, we shall have $x + y + z + w + u = 20$; and if from this the given equations are successively subtracted, the values of the unknown numbers become known.

8. $\begin{cases} x + y = 5. \\ y + z = 7. \\ z + x = 6. \end{cases}$

9. $\begin{cases} x + y + z = 6. \\ y + z + w = 9. \\ x + w + z = 8. \\ x + y + w = 7. \end{cases}$

10. $\begin{cases} \dfrac{x}{2} + y = 12. \\ \dfrac{y}{3} + z = 15. \\ \dfrac{z}{4} + x = 9. \end{cases}$

11. $\begin{cases} x + \frac{1}{2}\, y + \frac{1}{3}\, z = 10. \\ y + \frac{1}{2}\, x + \frac{1}{3}\, z = \ \ 8. \\ z + \frac{1}{3}\, x + \frac{1}{2}\, y = 12. \end{cases}$

5 12. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 7. \\ \dfrac{1}{y} + \dfrac{1}{z} = 9. \\ \dfrac{1}{x} + \dfrac{1}{z} = 8. \end{cases}$

NOTE. Subtract from half the sum of the three equations each equation successively.

13. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{2}. \\ \dfrac{1}{y} + \dfrac{1}{z} = \dfrac{5}{18}. \\ \dfrac{1}{x} + \dfrac{1}{z} = \dfrac{4}{9}. \end{cases}$

Ans. $\begin{cases} x = 3. \\ y = 6. \\ z = 9. \end{cases}$

14. $\begin{cases} \dfrac{6}{x} + \dfrac{5}{y} = 11 \\ \dfrac{3}{x} + \dfrac{4}{z} = 7. \\ \dfrac{10}{y} - \dfrac{2}{z} = 8. \end{cases}$

NOTE. In eliminating do not clear of fractions.

15. $\begin{cases} \dfrac{1}{x} - \dfrac{1}{y} + \dfrac{1}{z} = \dfrac{1}{3}. \\ \dfrac{1}{x} + \dfrac{1}{y} - \dfrac{1}{z} = \dfrac{2}{3}. \\ \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1. \end{cases}$

16. $\begin{cases} \dfrac{1}{x} - \dfrac{1}{y} - \dfrac{1}{z} = -1. \\ \dfrac{1}{y} - \dfrac{1}{x} - \dfrac{1}{z} = -3. \\ \dfrac{1}{z} - \dfrac{1}{x} - \dfrac{1}{y} = -5. \end{cases}$

PROBLEMS

PRODUCING SIMPLE EQUATIONS CONTAINING TWO OR MORE UNKNOWN NUMBERS.

27. Many of the problems given in Chapter III. contain two or more unknown numbers; but in every case these are so related to one another that if one becomes known the others become known also; and therefore the problems can be solved by the use of a single letter. But many problems, on account of the complicated conditions, cannot be performed by the use of a single letter. No problem can be solved unless the conditions given are sufficient to form as many independent equations as there are unknown numbers.

1. A farmer sold 10 barrels of apples and 3 barrels of potatoes for \$29; and at the same rate 4 barrels of apples and 5 barrels of potatoes for \$23. Find the price of each a barrel.

Let $x =$ the no. of dollars for a barrel of apples,
and $y =$ " " " " potatoes.
Then, by the conditions, $10x + 3y = 29$,
and $4x + 5y = 23$.
Solving these equations we have

$$x = 2 \quad \text{and } y = 3.$$

2. Find two numbers such that the greater exceeds three times the less by 6, and three times the greater exceeds the less by 22. Ans. $\frac{1}{2}$ and $7\frac{1}{2}$.

3. If the numerator of a fraction is one less, the resulting fraction is $\frac{1}{3}$; but if the denominator is two less, the resulting fraction is $\frac{1}{2}$. Find the fraction.

Let $\frac{x}{y} =$ the fraction.

Then $\frac{x-1}{y} = \frac{1}{3}$ and $\frac{x}{y-2} = \frac{1}{2}$.

Solving these equations, we have

$$x = 5 \quad \text{and } y = 12,$$

and the fraction is $\frac{5}{12}$.

4. Find two numbers such that three times the first, minus four times the second, is zero; and half the first, plus a third of the second, is 9. Ans. 12 and 9.

5. The ages of two persons, A and B, are such that 4 years ago A's age was three times B's, and 10 years hence A's age will be double B's. What is the age of each?

Ans. A's 46; B's 18.

6. Find a fraction such that if 4 is added to its numerator the resulting fraction is $\frac{7}{8}$; but if 4 is added to the denominator the resulting fraction is equal to $\frac{1}{4}$. Ans. $\frac{3}{8}$.

7. A number of two figures, whose sum is 8, is such that if 36 is subtracted from it the order of the figures is reversed. What is the number? Ans. 62.

Let $x =$ the figure in the tens' place,

and $y =$ " " units' place.

Then $10x + y =$ the number.

Then $$10x + y - 36 = 10y + x$$

and $$x + y = 8$$

Solving these equations we have $x = 6$, and $y = 2$.

8. A number of two figures, whose sum is 9, is such that if 27 is added to it the order of the figures is reversed. What is the number?

9. A man worked 9 days and his son 5 days, and they received for their work \$23. At another time the man worked 7 days and his son 5 days, and they received \$19. What were the wages of each?

10. A farmer who had \$55 in his pocket gave to each man of his laborers \$3, and to each boy \$1, and had \$8 left. If he had given each man \$3.50, and then each boy \$2, until he had given away all the money, 2 boys would have received nothing. How many men and how many boys did he hire?

11. As William and Henry were talking of their money, William said to Henry, "Give me 10 cents and I shall have 5 times as much as you will have left." Henry said to William, "Give me 10 cents and I shall have as much as you will have left." How many cents did each have?

Ans. William, 40 cents; Henry, 20 cents.

12. A and B began business with different sums of money. The first year A gains $250, and B loses $150, and then A's money is four-fifths of B's. If A had lost $150, and B gained $250, A's money would then have been one-half of B's. With what sum did each begin? Ans. A $1350; B $2150.

13. If 5 is added to the numerator of a certain fraction, the value will be a unit; and if 3 is subtracted from its denominator, the value will be $\frac{1}{2}$. What is the fraction?

14. Find three fractions such that the sum of the first and second is $\frac{9}{20}$, of the second and the third $\frac{11}{30}$, and of the first and third $\frac{5}{12}$.

15. There is a number of three figures whose sum is 6; the right-hand figure is equal to the sum of the other two, and if 198 is added to the number the order of the figures is reversed. Find the number. Ans. 123.

Let $x =$ the figure in units' place.
$y =$ " " " tens' place.
$z =$ " " " hundreds' place.

Then $100z + 10y + x =$ the number.

By the conditions $x + y + z = 6$

$$y + z = x$$

and $100z + 10y + x + 198 = 100x + 10y + z.$

Solving these equations we have $x = 3, \quad y = 2, \quad z = 1.$

16. Find three numbers such that the sum of the first and second is 85, of the second and third 142, of the third and first 87.

17 If I can row 6 miles down a river in an hour, while it takes me 3 hours to row back, at what rate can I row in still water, and at what rate does the water flow in the river?

Let $x =$ no. miles an hour in still water,
and $y =$ " " the water flows.

Then by the conditions $x + y = 6$ (1)

$x - y = \frac{6}{3} = 2$ (2)

Add (1) and (2) $2x = 8$

Subtract (2) from (1) $2y = 4$

$x = 4$

$y = 2$

18. A father wished to give to each of his children 25 cents, but found he needed 10 cents more than he had to do it; so he gave each 20 cents, and had 25 cents left. How many children, and how much money did he have? Ans. 7 children, and $1.65.

19. The year A. D. of the invention of printing by Gutenburg is expressed by a number of four figures whose sum is 14. The units' figure is double the tens', the thousands' figure is equal to the hundreds minus the tens; and if 4905 is added to the number, the order of the figures will be reversed. What is the date? Ans. 1436.

20. A boy paid 80 cents with 11 pieces of silver, 5 and 10 cent pieces. How many pieces of each did he give?

21. Peter said to John: "If you give me $1, I shall have twice as much as you have left." John said to Peter: "If you give me $1, I shall have three times as much as you have now." How much did each have?

22. A certain fraction becomes $\frac{3}{4}$ when its numerator is increased by 3, and its denominator is multiplied by 3; but if its denominator is increased by 3, and its numerator multiplied by 3, it becomes $\frac{9}{10}$. Find the fraction.

23. A certain fraction becomes $\frac{3}{5}$ by adding 1 to both numerator and denominator, but $\frac{1}{2}$ by subtracting 1 from both terms. Find the fraction.

24. A certain fraction is doubled by adding 6 to its numerator, and 9 to its denominator; and trebled by adding 2 to its numerator, and taking 3 from its denominator. Find the fraction.

25. A and B can do a piece of work together in $1\frac{1}{5}$ days. A and C in $1\frac{1}{3}$ days, and B and C in $1\frac{5}{7}$ days. Find the time in which each can do it alone.

26. A and B can do a piece of work together in $3\frac{3}{4}$ days. It can also be done if A works 5 days, and B 3 days. Find the time in which each can do it alone.

27. A and B can together do a piece of work in 20 days. They work together 15 days, and then A finishes the work alone in 15 days more. In what time can each do the work alone?

28. A and B together finished a piece of work in $2\frac{2}{9}$ days. If A had worked twice as fast, and B half as fast, it would have taken them $1\frac{2}{3}$ days. In how many days could each do the work alone?

29. The sum of the three figures of a number is 9; the tens' figure is 1; and if the order of the figures is reversed the number so formed exceeds the original number by 396. What is the number?

30. If I divide a certain number of two figures by the sum of its figures, the quotient is 7, and the remainder 6. If I reverse the order of the figures, and divide the resulting number by the difference of the figures, the quotient is 7, and the remainder 3. Find the number.

31. A number of three figures, whose sum is 12, has the right-hand figure zero. If the left-hand and middle figures change places the number is increased by 180. What is the number?

32. A man has \$900 at interest in three sums, the first at 4%, the second at 5%, and the third at 6%, receiving for interest \$47 a year. The part at 5% is half as much as the other two together. Find the three parts.

CHAPTER V.

ALGEBRAIC NUMBERS.

28. ALGEBRA is the science which treats of numbers. These numbers are represented by figures and letters. In general it is agreed that the first letters of the alphabet, a, b, c, etc., shall stand for what are called *known* numbers, that is, those whose values are *given;* and the last letters, x, y, z, etc., for *unknown* numbers, that is, those whose values are *to be determined.*

29. In Algebra numbers are of *two kinds, positive* and *negative.* Thus, on the scale of the thermometer, the numbers *above* the zero mark are called *positive,* before which no sign, or the $+$ sign, is placed; and those *below* the zero mark are called *negative,* before which the $-$ sign is placed. 80°, or $+$ 80°, indicates eighty degrees *above* zero, and $-$ 12° indicates twelve degrees *below* zero.

So the numbers representing north latitude, east longitude, gain, etc., are usually called positive, and their opposites, that is, south latitude, west longitude, loss, etc., are called negative. Negative numbers, as compared with positive numbers, mean opposite in direction, or in effect, and are as real as positive numbers. There is nothing in the nature of things to prevent our considering positive numbers as negative, and negative numbers as positive.

30. A clear understanding of these numbers in all their relations can best be obtained through the device of a straight line with the zero point at its centre, and positive numbers extending to the right, and negative numbers to the left, indefinitely, thus,

$$\ldots -5,\ -4,\ -3,\ -2,\ -1,\ \ 0,\ +1,\ +2,\ +3,\ +4,\ +5, \ldots$$

a series, embracing all possible integral numbers, which increase by one indefinitely from left to right, and decrease by one indefinitely from right to left. In this series every negative number is considered to be less than zero; and, in general, every number in the series is considered to be less than any number following it, and greater than any number preceding it, that is, *relatively* so.

In this *relative* sense the phrases *greater than*, *less than*, *less than zero*, are to be understood, unless the contrary is expressed.

Arithmetic takes into account that part only of the series to the right of the zero, while algebra makes use of the whole series. From this it will be seen at once, that, in *algebra*, addition, subtraction, multiplication, and division, covering as they do both positive and negative numbers, must have a more extended signification than in arithmetic.

CHAPTER VI.

ADDITION.

31. ADDITION in algebra is the process of finding the aggregate, or sum, of two or more algebraic numbers.

If the numbers to be added are all positive, the *algebraic sum* is *positive* and equal in amount to the number of positive units; if they are all negative, the algebraic sum is *negative* and equal in amount to the number of negative units; and if they are both positive and negative, the algebraic sum is *positive* if the positive units are in the excess, and is equal in amount to that excess, *negative* if the negative units are in the excess, and is equal in amount to that excess, and *zero* if the sums of the positive and negative units are equal. Thus,

$$\left.\begin{array}{ll}(1) & +5+(+3)=+8\\ (2) & -5+(-3)=-8\\ (3) & +5+(-3)=+2\\ (4) & -5+(+3)=-2\\ (5) & +5+(-5)=\ \ \ 0\end{array}\right\}\quad \cdots \quad (A)$$

32. To illustrate (1) in the group above, move a pencil from the zero point along the line of numbers in the positive direction

☞ + — ☜

$-5,\ -4,\ -3,\ -2,\ -1,\ \ 0,\ +1,\ +2,\ +3,\ +4,\ +5,$

(that is to the right) five spaces, then three spaces; the distance from the starting point is eight spaces to the right, and is represented by $+ 8\,s$, the *algebraic sum* of the distances moved.

To illustrate (2), move a pencil from the zero point along the line of numbers in the negative direction (that is to the left)

five spaces, then three spaces; the distance from the starting point is eight spaces to the left, and is represented by $-8s$, the *algebraic sum* of the distances moved.

☞ + — ☜

$-5, \ -4, \ -3, \ -2, \ -1, \ 0, \ +1, \ +2, \ +3, \ +4, \ +5,$

To illustrate (3), move a pencil from the zero point along the line of numbers in the positive direction five spaces, then in the negative direction three spaces; the distance from the starting point is two spaces to the right, and is represented by $+2s$, the *algebraic sum* of the distances moved.

To illustrate (4), move a pencil from the zero point along the line of numbers in the negative direction five spaces, then in the positive direction three spaces; the distance from the starting point is two spaces to the left, and is represented by $-2s$, the *algebraic sum* of the distances moved.

To illustrate (5), move a pencil from the zero point along the line of numbers in the positive direction five spaces, then in the negative direction five spaces; the distance from the starting point is zero, and is represented by $0s$, the *algebraic sum* of the distances moved.

33. The *algebraic sum* is not then, as in arithmetic, the entire number of spaces moved, but the distance at the cessation of the movement, from the starting point.

And, in general, the algebraic sum of several numbers is the deviation of the result from zero, the positive units being counted *on*, or employed to affect the result, according to their number, in one way, and the negative units being counted *off*, or employed to affect the result, according to their number, in the opposite way.

In a parliamentary body the members on one side may be supposed to represent positive numbers, and the members of the opposition negative numbers. With such a supposition the *algebraic sum* is the *majority*. Thus, if there are 87 on one side, and 69 in opposition, the majority is $87 - 69$, or 18;

that is, the majority is the sum of 87 and − 69. Pairing does not affect the result. Thus, if in the case above 2 on one side had paired with 2 in opposition, the numbers would stand 85 and − 67, and the majority would have been 85 − 67, or 18, as before.

34. Illustrative Problems.

1. Suppose a man to walk along a straight road 70 rods forward and then 50 rods backward, his distance from his starting point is 20 rods.

But if he first walks 50 rods forward, and then 70 rods backward, his distance from his starting point would be 20 rods, but *on the opposite side of his starting point.*

The corresponding algebraic statements are

$$70 \text{ rd.} + (-50 \text{ rd.}) = +20 \text{ rd.}$$
$$50 \text{ rd.} + (-70 \text{ rd.}) = -20 \text{ rd.}$$

2. Suppose that I have a house worth \$5000, and a piece of land worth \$2000, and that I owe \$500; then the net value of my property is \$5000 + \$2000 + (− \$500) = + \$6500. Again, suppose my house is worth \$5000, my land \$2000, while I owe \$8000; then the net value of my property is \$5000 + \$2000 + (− \$8000) = − \$1000; that is, I am worth − \$1000, or, in other words, I owe \$1000 more than I can pay.

3. A merchant was engaged in business for two years; the first year he gained \$2000, and the second he gained − \$500 (that is, he lost \$500). How did he come out at the end of the second year? Ans. With a gain of \$1500.

4. A thermometer indicated + 50° (50° above 0); it then fell 20°, then rose 40°. What temperature did it then indicate? Had it fallen 40°, instead of risen 40°, what would have been the temperature? Ans. 70°; − 10°.

5. A balloon ascending from Boston was driven due east 30 miles the first hour, and 25 the second, when, rising higher,

it encountered a wind which swept it due west at the rate of 40 miles an hour. Required, its position at the end of 4 hours from the start.

Ans. 25 miles west of Boston; or, -25 miles from Boston.

In this example which contains the greater number of units, the arithmetical or the algebraic sum?

Ans. The arithmetical, by 110 units.

6. A ship sailing at the rate of 6 miles an hour, sails three hours due east and one hour due west. How far is she from her starting point?

7 If a ship sails at the rate of 10 miles an hour three hours due east, and four hours due west, how far is she from her starting point?

8. A man earns \$3 a day, and runs up a store-bill of \$4 a day. Write down his standing at the end of 6 days.

DEFINITIONS.

35. An **Algebraic Expression** is a single number, or a collection of numbers, generally connected by algebraic signs.

36. A **Monomial** is an algebraic expression which contains a single term; as, a, or $4\,x$, or $7\,c\,x\,z$.

37. A **Polynomial** is an algebraic expression which contains two or more terms; as, $a + b$, $5\,x + 3\,y - 6\,b\,x\,y$, or $b + 3\,c - 4\,d + e$.

38. A **Binomial** is a polynomial of two terms; as, $2\,a + 5\,y$, or $a - b$.

39. A **Trinomial** is a polynomial of three terms; as, $a + 5\,x - 7\,a\,b$.

40. **Like Terms**, or **Similar Terms**, are those which do not differ, or differ only in their signs or coefficients; as, $5\,a\,b$, and $-3\,a\,b$. Other terms are *unlike* or *dissimilar;* as, $7\,x\,y$ and $6\,b\,y$.

ADDITION OF ALGEBRAIC LITERAL EXPRESSIONS.

CASE I.

41. To find the Sum of Monomials when they are Similar and have Like Signs.

1. William has 7 apples, James 5 apples, and Henry 4 apples; how many apples have they all?

7 apples,		$7a$
5 apples,	or letting a	$5a$
4 apples,	represent	$4a$
	one apple,	
16 apples,		$16a$

It is evident that just as 7 apples and 5 apples and 4 apples added together make 16 apples, so $7a$ and $5a$ and $4a$ added together make $16a$.

In the same way $-7a$ and $-5a$ and $-4a$ are equal together to $-16a$.

Therefore, when the monomials are similar, and have like signs, we have the following

Rule.

Add the coefficients, and to their sum annex the common letter or letters, and prefix the common sign.

(2.)	(3.)	(4.)	(5.)	(6.)	(7.)
$2by$	$6xy$	$3az$	$4a$	$-6z$	$-6ax$
$5by$	$2xy$	az	a	$-5z$	$-5ax$
$3by$	$5xy$	$7az$	$7a$	$-9z$	$-ax$
$7by$	$7xy$	$2az$	$2a$	$-z$	$-4ax$
$17by$		$13az$	14	$-21z$	

8. What is the sum of x, $5x$, $7x$, and $2x$? Ans. $15x$.
9. What is the sum of $3bx$, $6bx$, $5bx$, and $2bx$?
10. What is the sum of $5ax$, $6ax$, $11ax$, and $8ax$?
11. What is the sum of $-xy$, $-7xy$, $-2xy$, and $-xy$? Ans. $-11xy$.
12. What is the sum of $-2bc$, $-5bc$, $-4bc$, and $-6bc$?
13. What is the sum of $-2bcd$, $-5bcd$, and $-3bcd$?

Case II.

42. To find the Sum of Monomials when they are Similar and have Unlike Signs.

1. A man earns 9 dollars one week, and the next week earns nothing and spends 4 dollars, and the next week earns 7 dollars, and the fourth week earns nothing and spends 3 dollars; how much money has he left at the end of the fourth week?

If what he earns is indicated by +, then what he spends will be indicated by —, and the example will appear as follows:

$$\left.\begin{array}{r} +\ 9 \text{ dollars,} \\ -\ 4 \text{ dollars,} \\ +\ 7 \text{ dollars,} \\ -\ 3 \text{ dollars,} \\ \hline +\ 9 \text{ dollars,} \end{array}\right\} \begin{array}{c}\text{or, letting } d \\ \text{represent} \\ \text{one dollar,}\end{array} \left\{\begin{array}{r} +\ 9\,d \\ -\ 4\,d \\ +\ 7\,d \\ -\ 3\,d \\ \hline +\ 9\,d \end{array}\right.$$

Earning 9 dollars and then spending 4 dollars, the man would have 5 dollars left; then earning 7 dollars, he would have 12 dollars; then spending 3 dollars, he would have left 9 dollars. Or he earns in all 9 dollars + 7 dollars = 16 dollars; and spends 4 dollars + 3 dollars = 7 dollars; and therefore has left the difference between 16 dollars and 7 dollars = 9 dollars; hence the sum of $+ 9\,d$, $- 4\,d$, $+ 7\,d$, and $- 3\,d$, is $+ 9\,d$.

Therefore, when the terms are similar, and have unlike signs, we have the following

Rule.

Find the difference between the sum of the coefficients of the positive terms, and the sum of the coefficients of the negative terms, and to this difference annex the common letter or letters, and prefix the sign of the greater sum.

(2.)	(3.)	(4.)	(5.)
x	x	$15\,ac$	$12\,xyz$
$3x$	$-3x$	$6\,ac$	$-14\,xyz$
$-x$	$7x$	$-7\,ac$	xyz
$7x$	$-5x$	$-ac$	$-12\,xyz$
$-4x$	$13x$	$3\,ac$	$15\,xyz$
$6x$		$16\,ac$	

(6.)	(7.)
$25\,bcd$	$8\,(a+b)$
$-50\,bcd$	$-7\,(a+b)$
$10\,bcd$	$5\,(a+b)$
$-48\,bcd$	$-3\,(a+b)$
$7\,bcd$	$-2\,(a+b)$
$-56\,bcd$	$(a+b)$

8. Find the sum of $7\,x$, $-14\,x$, $13\,x$, and $-3\,x$.

9. Find the sum of $8\,(a+b)$, $11\,(a+b)$, $-(a+b)$, and $4\,(a+b)$. Ans. $22\,(a+b)$.

10. Find the sum of $-5\,az$, $+5\,az$, $+10\,az$, $+25\,az$, and $-13\,az$.

11. Find the sum of $27\,bc$, $-33\,bc$, $-14\,bc$, $28\,bc$, and $-12\,bc$.

12. Find the sum of $-14\,bc$, bc, $11\,bc$, $-bc$, $13\,bc$, and $-bc$.

13. Find the sum of $(b+c)$, $-(b+c)$, $12\,(b+c)$, and $-10\,(b+c)$. Ans. $2\,(b+c)$.

Case III.

43. To find the Sum of any Monomials.

From (3) in group (A) page 43, it follows that

$$a + (-b) = a - b$$

and that the sum of a, $-b$, and $-c$, or

$$a + (-b) + (-c) = a - b - c$$

No further reduction is possible, and therefore, to add dissimilar monomials we have the following

Rule.

Write them one after the other, each with its proper sign.

All algebraic expressions can be so written, and the result, without further reduction, is sometimes called an *algebraic sum.*

44. It should be remarked that

$$a+b+c=a+c+b=b+a+c=\text{etc.}$$

that is, the sum of any number of algebraic expressions is *independent of their order.*

45. Further, $\quad +(a+b)=+a+b$

For, to put a and b together, and then add the result to what has gone before, is the same as to add both a and b to what has gone before. Similarly,

$$a+(b+c)=(a+b)+c$$

that is, the sum of any number of algebraic expressions is *independent of the mode of grouping them.*

46. It follows that

the sum of $4\,a$ and $5\,b$ is neither $9\,a$ nor $9\,b$, and can only be expressed in the form of $4\,a+5\,b$, or $5\,b+4\,a$; and the sum of $7\,a$ and $-3\,b$ is $7\,a-3\,b$, or $-3\,b+7\,a$. In finding the sum of $4\,a$, $5\,b$, $7\,a$, and $-3\,b$, the a's can be added together by Case I., and the b's by Case II., and the two results arranged according to §§ 44, 45; thus, $4\,a+5\,b+7\,a-3\,b=4\,a+7\,a+5\,b-3\,b=11\,a+2\,b$, or $2\,b+11\,a$, regardless of the order of the terms.

1. Find the sum of $6\,b$, $-3\,c$, $5\,a$, $4\,y$, $3\,a$, $-5\,c$, $7\,a\,b$, $4\,b$, $6\,a$, $8\,c$, $-4\,y$, and $-4\,a\,b$.

$$\begin{array}{rrrrr}
6b & -3c & +5a & +4y & +7ab \\
4b & -5c & +3a & -4y & -4ab \\
 & +8c & +6a & & \\
\hline
10b & & +14a & & +3ab
\end{array}$$

For convenience, similar terms are written under each other; then by Case I. the first column at the left is added; the second by Case II., and so on.

Therefore, to find the sum of any monomials, we have the following

Rule.

Add the similar terms according to Cases I. and II., and write after these results, in any order, the dissimilar ones, each with its proper sign.

2. Find the sum of $6a$, $-5b$, $+3c$, $+4b$, $-7c$, $+6d$, $-3c$, $+8a$, and $-5d$.

3. $9+(-7)+(-5)+0+(+8)+(-4)=?$

4. Find the sum of $5a$, $-6b$, $+4x$, $+4b$, $-3z$, $+5x$, $+a$, $+3b$, $-4z$, $+3y$, $-5a$, $+8b$, $-a$, $-4c$, $-7x$, and $+8z$.

5. $3x+(-5y)+(+6x)+(+8y)+(-4y)=?$

6. $4a+(-5b)+(+6a)-(-2b)+(-8a)+(+7b)=?$

CASE IV.

47. To find the Sum of Polynomials.

Polynomials are groups of monomials, and hence everything necessary to a complete understanding of their addition has been explained in the three foregoing cases.

We have, therefore, for the addition of polynomials the following

Rule.

Write similar terms under each other, find the sum of each column, and connect the several sums with their proper signs.

1. Find the sum of

$$3a+2b-c,\ -a+3b+2c,\ 2a-3b+5c.$$

$$\begin{array}{r} 3a+2b-c \\ -a+3b+2c \\ 2a-3b+5c \\ \hline 4a+2b+6c \end{array}$$

2. Add

$$-5ab+8bc-9ac,\ 8ab-4bc+3ac,\ -2ab-2bc+2ac.$$

$$\begin{array}{r} -5ab+8bc-9ac \\ 8ab-4bc+3ac \\ -2ab-2bc+2ac \\ \hline ab+2bc-4ac \end{array}$$

3. Add $2x - 3ax + 1$, $5x - ax + 5$, $x + 2ax - 8$.

$$\begin{array}{r} 2x - 3ax + 1 \\ 5x - ax + 5 \\ x + 2ax - 8 \\ \hline 8x - 2ax - 2 \end{array}$$

Add the following :

4. $4a + 3b + 5c$, $-2a + 3b - 8c$, $a - b + c$.
Ans. $3a + 5b - 2c$.

5. $5a - 2b - 6c$, $3a - 5b + 2c$, $-2a + 3b - c$.
Ans. $6a - 4b - 5c$.

6. $-15a - 19b - 18c$, $14a + 15b + 8c$, $a + 5b + 9c$.
Ans. $b - c$.

7. $25a - 15b + c$, $13a - 10b + 4c$, $a + 20b - c$.
Ans. $39a - 5b + 4c$.

8. $4a + 2b + 8$, $-2a - 7b - 13$, $-7a - 15b - 7$, $3a + 4b - 11$.

9. $2a - 3b + c$, $15a - 21b - 8c$, $3a + 24b + 7c$.
Ans. $20a$.

10. $23a - 17b - 2c$, $-9a + 15b + 7c$, $-13a + 3b - 4c$.
Ans. $a + b + c$.

11. $a + 2b - c$, $2a - b + 2c$, $-3a - b - c$.

12. $2a - 5b + 2c$, $2b - 5c + 2a$, $2c - 5a + 2b$.
Ans. $-a - b - c$.

13. $3x + 7 - 5x$, $2x - 8 - 9a$, $4a - 2x + 3x$.
Ans. $x - 5a - 1$.

14. $5a - 3c + d$, $b - 2a + 3d$, $4c - 2a - 3d$.
Ans. $a + b + c + d$.

15. $5a - 3ab + 9ab + 17b$, $-2a + 6ab - 4ab - 12b$, $b - 4ab - 5ab - a$, $2ab - 2a - 6b - ab$.

16. $a - 4ab + 6abc$, $ab - 10abc + c$, $b + 3ab + abc$.
Ans. $a + b + c - 3abc$.

17. $6a + 7b + 11c + 4d + 9$, $8c - 5b + 2a - 11d - 7$, $3a - 5b - 9c + 6$, $d + a + 10b - 3c$, $-7a - 6b - 7c - 5d - 10$.
Ans. $5a + b - 11d - 2$.

CHAPTER VII.

SUBTRACTION.

48. Subtraction is the process of finding the *difference* between two numbers. This *difference* is the number of units which lie between the two numbers, or is what must be added to the subtrahend to produce the minuend.

Subtrahend + Remainder = Minuend.

49. Thus, in the group below, considering the 5's as minuends and the 3's as subtrahends, by determining what must be added to the subtrahend to produce the minuend, that is, by addition, we obtain the following:

$$\left.\begin{array}{ll} (1) & +5-(-3)=+8 \\ (2) & -5-(+3)=-8 \\ (3) & +5-(+3)=+2 \\ (4) & -5-(-3)=-2 \end{array}\right\} \quad . \; . \; . \quad (B)$$

Wherever the points indicated by the minuend and subtrahend are situated in the series of numbers, a certain number of spaces must be between them, and this number with the appropriate sign represents their *algebraic difference*. If the movement is to the right in going from the point indicated by the subtrahend to the point indicated by the minuend, the + sign must be prefixed; if to the left, the − sign.

To illustrate (1), in the group above, move a pencil from the point three spaces to the left of 0 to a point five spaces to the

☛ + — ☚

$-5,\ -4,\ -3,\ -2,\ -1,\ \ 0,\ +1,\ +2,\ +3,\ +4,\ +5,$

right of 0; the distance moved is eight spaces in the positive direction; hence the *algebraic difference* is + 8 *s*.

To illustrate (2), move a pencil from the point three spaces to the right of 0 to a point five spaces to the left of 0; the distance moved is eight spaces in the negative direction; hence, the *algebraic difference* is $-8s$.

+ −

$-5,\ -4,\ -3\ \ -2,\ -1,\ \ \ 0,\ +1,\ +2,\ +3,\ +4,\ +5.$

To illustrate (3), move a pencil from the point three spaces to the right of 0 to a point five spaces to the right; the distance moved is two spaces in the positive direction; hence, the *algebraic difference* is $+2s$.

To illustrate (4), move a pencil from a point three spaces to the left of 0 to a point five spaces to the left of 0: the distance moved is two spaces in the negative direction; hence, the *algebraic difference* is $-2s$.

50. From groups (B) and (A), pp. 53, 43, (2) and (1,) it follows that subtracting a positive number is equivalent to adding an equal negative number, and subtracting a negative number is equivalent to adding an equal positive number.

Therefore, to subtract one number from another, we have the following

Rule.

Change the sign of the subtrahend and proceed as in addition.

51. Illustrative Problems.

1. Suppose I am worth \$7000; it matters not whether a thief steals \$3000 from me, or a rogue having the authority involves me in debt \$3000 for a worthless article; for in either case I shall be worth only \$4000. The thief *subtracts* a *positive* quantity; the rogue *adds* a *negative* quantity.

The corresponding algebraic statements are

$$\$7000 - (+\$3000) = +\$4000$$

and

$$\$7000 + (-\$3000) = +\$4000$$

2. The Roman emperor, Claudius I., was born B. C. 10, and died A. D. 54. How old was he when he died?

3. If A has $600 and no debts, and B has no property but owes $200, how much better off is A than B? Ans. $800.

4. The longitude of Berlin is 13° E., that of Boston 71° W. What is their difference of longitude?

5. The longitude of St. Louis is 90° W., that of Boston 71° W. What is their difference of longitude?

Subtraction of Algebraic Literal Expressions.

52. From group (B) we learn that, in general, *the subtraction of a positive number is equivalent to adding an equal negative number, and the subtraction of a negative number is equivalent to adding an equal positive number.*

Therefore, for the subtraction of monomials and polynomials, we have the following

Rule.

Change the sign of each term of the subtrahend from + to —, or — to +, or suppose each to be changed, and then proceed as in addition.

	(1.)	(2.)	(3.)	(4.)	(5.)	(6.)	(7.)
Min.	9	9	9	9	9	9	9
Sub.	9	6	3	0	-3	-6	-9
Rem.	0	3	6	9	12	15	18

In examples 1–7, the minuend remaining the same while the subtrahend becomes in each 3 less, the remainder in each is 3 greater than in the preceding.

	(8.)	(9.)	(10.)	(11.)	(12.)	(13.)	(14.)
Min.	9	6	3	0	-3	-6	-9
Sub.	9	9	9	9	9	9	9
Rem.	0	-3	-6	-9	-12	-15	-18

In examples 8–14, the minuend in each becoming 3 less while the subtrahend remains the same, the remainder in each is 3 less than in the preceding.

	(15.)	(16.)	(17.)	(18.)	(19.)	(20.)	(21.)
Min.	9	6	3	0	-3	-6	-9
Sub.	9	6	3	0	-3	-6	-9
Rem.	0	0	0	0	0	0	0

In examples 15–21, both minuend and subtrahend decreasing by 3, the remainder remains the same.

	(1.)	(2.)	(3.)	(4.)	(5.)
Min.	$9a$	$10x$	$-7y$	$-12z$	$11cd$
Sub.	$5a$	$-3x$	$-3y$	$+7z$	$3cd$
Rem.	$4a$	$13x$	$-4y$	$-19z$	

	(6.)	(7.)
Min.	$5a+3b-7c$	$4a-3b-5c$
Sub.	$3a+2b-5c$	$6a+5c-4d$
Rem.	$2a+b-2c$	$-2a-3b-10c+4d$

8. From $12a$ subtract $3a$.

9. From $-17ab$ subtract $-3ab$.

10. From $10a-6b-8c$ take $4a+2b-3c$.

11. From $3x+6y+11z$ take $3y-3z$.

12. From $7x-12y-12z$ take $6x-8y+3z$.

13. From $3ab+4cd-5ac-6bd$ take $3ab+2cd-3ac+3bd$.

14. Subtract $5x+3y-1$ from $6x$, and add the result to $7x-5y+2$.

15. What must be added to $3a-4b+c$ to produce $5a+3b-2c$.

16. What must be subtracted from $10a-3b-2c+d$ to leave $3a+5b-c-d$.

17. To what must $3a-4b+c$ be added to produce zero?

18. What must be added to $7a-4b+11$ to give $11a-8$?

19. From what must the sum of $8a-3$, $3a-4$, and $4a+6$ be subtracted to give $5a-25$?

Removal and Introduction of Brackets.

53. The addition or subtraction of a polynomial may be indicated by enclosing the polynomial in a bracket and prefixing the sign + for addition, and the sign − for subtraction. The polynomial $d-e+f$ added to the polynomial $a+b-c$, that is, $a+b-c+(d-e+f)$, equals *by the rules of addition* $a+b-c+d-e+f$; and the polynomial $d-e+f$ subtracted from the polynomial $a+b-c$, that is, $a+b-c-(d-e+f)$, equals *by the rules of subtraction* $a+b-c-d+e-f$.

Therefore, for the removal of brackets we have the following

Rule.

When the bracket with the plus sign before it is removed, the included terms must be rewritten without change of sign; but when a bracket with the minus sign before it is removed, the included terms must be rewritten with change of sign.

54. When there are several brackets, they may be removed *one at a time*.

Thus,
$$\begin{aligned} &4x-\{3x-(2x-\overline{x-a})\}\\ &=4x-\{3x-(2x-x+a)\}\\ &=4x-\{3x-2x+x-a\}\\ &=4x-3x+2x-x+a\\ &=2x+a \end{aligned}$$

In the above process the vinculum was removed first, and then the brackets in succession, beginning with the inner one.

If we remove the outer bracket first, the work will appear as follows;

$$4x-3x+(2x-\overline{x-a}) \quad (1)$$
$$=4x-3x+2x-x+a \quad (2)$$

As in (1) the + sign appears before the bracket, this might have been omitted at once without change of signs, and at the same time the *vinculum* over $x-a$ omitted, and the final expression obtained at once. Thus from (1) we have at once $4x-3x+2x-x+a=2x+a$

While the beginner may secure accuracy by the longer method, it is recommended that the more advanced student begin with the outermost bracket, as the shorter method.

Remove the brackets and reduce each of the following expressions to its simplest form :

1. $a + b + (3a - 2b)$. 2. $a + b - (a - 3b)$.
3. $4a - \{3a - (2a - a)\}$.
4. $x + [2x - \{3x + (4x - 6x)\}]$.
5. $5 - [4 + \{3 - (2 - x)\}]$.
6. $a - \{a - (a - \overline{a - b})\}$.
7. $7a - [5b - \{4a - (8a - 2b)\}]$.
8. $a - (b - c) - \{b - (a - c) - [a - \{2b - (a - c)\}]\}$.

55. Conversely, from § 52, we have for the introduction of brackets the following

Rule.

When the bracket introduced is preceded by the plus sign, all the terms enclosed must be written without change of sign; but when the bracket is preceded by a minus sign, all the terms enclosed must be written with change of sign.

According to this rule a polynomial can be written in a variety of ways.

Thus,
$$\begin{aligned} & a + b + c - d + f \\ = & a + (b + c - d + f) \\ = & a - (-b - c + d - f) \\ = & a + b + c - (d - f) \\ = & a + b - (-c + d - f) \\ = & \text{etc.} \end{aligned}$$

Place in brackets, with the sign $-$ prefixed, without changing the value of the expression :

1. The last two terms of $a + b - c + d$.
2. The last three terms of $a - b - c - d$.
3. The first three and the last three terms of $-a - b + c - d + e - f$.
4. The last four terms of $-a - b - c + d + e$.

CHAPTER VIII.

MULTIPLICATION.

56. MULTIPLICATION is a short method of finding the sum of the repetitions of a number.

57. A **Factor** is any one of several numbers which are to be multiplied together to form a *product.*

58. Any one or more of the factors which go to make up the product may be called the **Coefficient** of the remaining factors. Thus, in $5abc$, 5 is the coefficient of abc, or bc the coefficient of $5a$, or $5ab$ the coefficient of c, and so on.

A coefficient is called numerical, literal, or mixed, according as it is a numeral, a letter or letters, or a numeral and letters combined. The three cases above, taken in their order, illustrate this.

By coefficient, the numerical coefficient, together with the sign of the expression, is usually meant. If no figure is expressed, a unit is understood; thus, x is the same as $1x$.

59. The **Reciprocal** of a number is a unit divided by that number; thus, the reciprocal of 3 is $\frac{1}{3}$; of x, $\frac{1}{x}$.

60. A **Power** is the product obtained by repeating a number a given number of times.

61. An **Index,** or **Exponent,** is a number either positive or negative, integral or fractional, placed at the right, and a little above the number.

If the index is *positive* and *integral,* it indicates how many times the number enters as a factor into the power.

Thus, $2^4 = 2 \times 2 \times 2 \times 2 = 16$; read, 2 fourth power, or the fourth power of 2.

$a^2 = a \times a$; read, a second power, or a square.

$a^3 = a \times a \times a$; read, a third power, or a cube.

$a^n = a \times a \times a \ldots$ to n factors; read, a nth power.

Exponents and coefficients must be carefully distinguished.

Thus, $$x^4 = x \times x \times x \times x$$
while $$4x = x + x + x + x$$

62. $ab = ba$; and $abc = acb = bca$; or the product of any number of factors is *independent of their order.*

63. Moreover $a(bc) = (ab)c = b(ca)$; that is, the product of any number of factors is *independent of the order of grouping them.*

64. If d is a positive integer, then $(a + b + c)d =$

$$(a + b + c) + (a + b + c) + (a + b + c) \ldots$$

repeated d times,

$$\begin{aligned} &= a + a + a + \ldots \text{ repeated } d \text{ times,} \\ &+ b + b + b + \ldots \quad \text{“} \quad \text{“} \\ &+ c + c + c + \ldots \quad \text{“} \quad \text{“} \\ &= ad + bd + cd; \end{aligned}$$

that is, the product of the sum of any number of algebraic numbers by a third is equal to the sum of the products obtained by multiplying the numbers separately by the third.

65. The multiplier must always be an abstract number, and the product is always of the *same nature* as the multiplicand.

The cost of 7 pounds of tea at 60 cents a pound is 60 cents taken, not 7 pounds times, but 7 times; and the product is of the same denomination as the multiplicand 60, viz., cents.

In Algebra the sign of the multiplier shows whether the repetitions are to be added or subtracted.

1. $$(+a) \times (+4) = +4a;$$
that is, $+a$ added 4 times is $+a+a+a+a=+4a$.

2. $$(+a) \times (-4) = -4a;$$
that is, $+a$ subtracted 4 times is $-a-a-a-a=-4a$.

3. $$(-a) \times (+4) = -4a;$$
that is, $-a$ added 4 times is $-a-a-a-a=-4a$.

4. $$(-a) \times (-4) = +4a;$$
that is, $-a$ subtracted 4 times is $+a+a+a+a=+4a$.

In the first and second examples the *nature* of the product is $+$; in the first, the $+$ sign of 4 shows that the product is to be added, and $+4a$ added is $+4a$; in the second, the $-$ sign of 4 shows that the product is to be subtracted, and $+4a$ subtracted is $-4a$. In the third and fourth examples the *nature* of the product is $-$; in the third, the $+$ sign of 4 shows that the product is to be added, and $-4a$ added is $-4a$; in the fourth, the $-$ sign of 4 shows that the product is to be subtracted, and $-4a$ subtracted is $+4a$.

66. Hence, in multiplication, we have for the sign of the product the following

Rule.

Like signs give $+$; *unlike*, $-$.

Hence the product of an *even* number of negative factors is positive; of an *odd* number, negative.

MULTIPLICATION OF ALGEBRAIC LITERAL EXPRESSIONS.

Case I.

67. When the Factors are Monomials.

1. Multiply $2a$ by $5b$.

$$2a \times 5b = 2 \times a \times 5 \times b = 2 \times 5 \times a \times b = 10ab$$

As the product is the same in whatever order the factors are arranged, we have simply changed their order and united in one product the numerical coefficients.

2. Multiply a^3 by a^2.

As the exponent, if integral and positive, of a number shows how many times it is taken as a factor,

$$a^3 = a \times a \times a$$

and

$$a^2 = a \times a$$

$$\therefore\ a^3 \times a^2 = (a \times a \times a) \times (a \times a) = a \times a \times a \times a \times a = a^5$$

Therefore the product of powers of the same number is *that number with an index equal to the sum of the indices of the factors.* This is called the Index Law.

Hence, when the factors are monomials, we have the following

Rule.

Annex the product of the literal factors to the product of their coefficients, remembering that like signs give +, and unlike, −.

(3)	(4.)	(5.)	(6.)	(7.)
$5x$	$5a^3b$	$6xy$	$-21yz^2$	$-2x^3y^4$
$2y$	$4ab^2$	$-3x^2y^2$	$4by^2$	$-7x^2y^2$
$10xy$	$20a^4b^3$	$-18x^3y^3$	$-84by^3z^2$	$14x^5y^6$

8. $a^3 \times a^4 = ?$

9. $x^2 \times x^5 = ?$

10. $c^3 \times (-c^2) = ?$

11. $-x^3 \times x^5 = ?$

12. $-a^7 \times (-a^6) = ?$

13. $4a^m \times 2a^n = ?$

14. $a^m \times a^n = ?$

15. $10x^2y^3 \times 4x^3y^4 = ?$

16. $-7a^2b^3 \times 5a^3b^5 = ?$

17. $-4x^4y^3 \times (-3x^3y^5) = ?$

18. $12a^2xy \times 4a^3x^2y^2 = ?$

19. $-12x^3z^2 \times (-3x^2y^3z) = ?$

20. $40\,a^3b^4 \times (-5a^4b^5) = ?$

21. $50\,a^2b^3c^2 \times 6a^3bc^4 = ?$

22. $5\,a\,b \times 4\,b\,c \times 3\,a\,b = ?$

23. $6\,a^2\,c^4\,d^3 \times 3\,b^2\,c^4 \times (-5\,a^4\,b^2) = ?$

24. $7\,x^3\,y\,z^2 \times (-2\,x^2\,y^2\,z^2) \times (-3\,x^4\,y^3\,z^2) = ?$

25. $2\,(x+y) \times 6\,(x+y)^2 = ?$

Case II.

68. When only one Factor is a Monomial.

1. Multiply $a + b + c$ by x.

$$(a + b + c)\,x = a\,x + b\,x + c\,x$$

or,

$$\begin{array}{r} a + b + c \\ x \\ \hline a\,x + b\,x + c\,x \end{array}$$

These results follow from § 64.

Therefore, for the multiplication of a polynomial by a monomial, we have the following

Rule.

Multiply each term of the multiplicand by the multiplier, and connect the several results by their proper signs.

(2.)

$$\begin{array}{r} 3\,x^2 + 5\,x - 7\,y \\ 5\,x\,y \\ \hline 15\,x^3\,y + 25\,x^2\,y - 35\,x\,y^2 \end{array}$$

3. $(a^3 + 3\,a^2 + 4\,a) \times 2\,a = ?$

4. $(a^2 - 2\,a\,b + b^2) \times a\,b = ?$

5. $(3\,x^3 - 5\,x^2 + 2\,x) \times 5\,x^2 = ?$

6. $(a^2 - b^2 - c^2) \times a\,b\,c = ?$

7. $(2\,x^4 - 3\,x^3 - x) \times (-5\,x^3) = ?$

8. $(5\,x^2\,y - 6\,x\,y^2 + 8\,x^2\,y^2) \times 3\,x\,y = ?$

9. $(a^3 - 3\,a^2\,b^2 + b^3) \times 3\,a^2\,b = ?$

10. $(8\,x^5\,y^{10} - 7\,x^3\,y^6 + 4\,x^2\,y^4 - 3\,x\,y^2) \times (-2\,x\,y^2) = ?$

11. $(x^3 - 3\,x^2\,y + 3\,x\,y^2 - y^3) \times x^2\,y = ?$

12. $(-5\,x\,y^2\,z + 3\,x\,y\,z^2 - 8\,x^2\,y\,z) \times x\,y\,z = ?$

Case III.

69. When the Factors are Polynomials.

1. Multiply $a + b + c$ by $x + y$.

$$(a + b + c)\,(x + y) = a\,(x + y) + b\,(x + y) + c\,(x + y)$$

$(x + y)$ is here regarded as a single term. The last expression further reduced becomes $a\,x + a\,y + b\,x + b\,y + c\,x + c\,y$, and this equals $a\,x + b\,x + c\,x + a\,y + b\,y + c\,y$.

Hence, for the multiplication of a polynomial by a polynomial, we have the following

Rule.

Multiply each term of the multiplicand by each term of the multiplier, and find the sum of the several products.

2. Multiply $3\,a^2 - 2\,a\,b + 4\,b^2$ by $2\,a - 3\,b$.

$$\begin{array}{rrrr}
3\,a^2 - & 2\,a\,b + & 4\,b^2 & \\
2\,a - & 3\,b & & \\
\hline
6\,a^3 - & 4\,a^2\,b + & 8\,a\,b^2 & \\
- & 9\,a^2\,b + & 6\,a\,b^2 - & 12\,b^3 \\
\hline
6\,a^3 - & 13\,a^2\,b + & 14\,a\,b^2 - & 12\,b^3
\end{array}$$

We begin at the left, placing the second result one place to the right, so that like terms may stand in the same vertical column.

3. Multiply $2x - 5x^2 + 2 + x^3$ by $2x^2 + 2 - x$.

$$
\begin{array}{rrrrrr}
x^3 - & 5x^2 + & 2x + & 2 & & \\
2x^2 - & x + & 2 & & & \\
\hline
2x^5 - 10x^4 + & 4x^3 + & 4x^2 & & & \\
- x^4 + & 5x^3 - & 2x^2 - & 2x & & \\
 & 2x^3 - & 10x^2 + & 4x + & 4 & \\
\hline
2x^5 - 11x^4 + & 11x^3 - & 8x^2 + & 2x + & 4 &
\end{array}
$$

In Examples 2 and 3 the multiplicand and multiplier were arranged according to the descending powers of x before multiplying. The polynomials could have been arranged according to the ascending powers as well.

4. $(x + a) \times (x + b) = ?$

5. $(2a + b) \times (2a + 3b) = ?$

6. $(3a^2 - 4a + 5) \times (2a + 5) = ?$

7. $(a^2 + ab + b^2) \times (a^2 - ab + b^2) = ?$

8. $(x^2 - x - 1) \times (x - 1) = ?$ Ans. $x^3 - 2x^2 + 1$.

9. $(a^2 - 2ax + 4x^2) \times (a^2 + 2ax + 4x^2) = ?$

10. $(a^2x - ax^2 + x^3 - a^3) \times (x + a) = ?$ Ans. $x^4 - a^4$.

11. $(16a^2 + 12ab + 9b^2) \times (4a - 3b) = ?$

12. $(x^2 + x - 2) \times (x^2 + x - 6) = ?$
Ans. $x^4 + 2x^3 - 7x^2 - 8x + 12$.

13. $(2x^3 - 3x^2 + 2x) \times (2x^2 + 3x + 2) = ?$

14. $(x^3 - 3x^2 + 3x - 1) \times (x^2 + 3x + 1) = ?$
Ans. $x^5 - 5x^3 + 5x^2 - 1$.

15. $(-a^5 + a^4b - a^3b^2) \times (-a - b) = ?$

16. $(a^3 + 2a^2b + 2ab^2) \times (a^2 - 2ab + 2b^2) = ?$
Ans. $a^5 + 4ab^4$.

17. $(x^2 - 3xy - y^2) \times (-x^2 + xy + y^2) = ?$

18. $(b^3 - a^2b^2 + a^3) \times (a^3 + a^2b^2 + b^3) = ?$
Ans. $a^6 - a^4b^4 + 2a^3b^3 + b^6$.

19. $(x^2 - 2xy + y^2) \times (x^2 + 2xy + y^2) = ?$

20. $(a^3 - 3a^2b + 3ab^2 - b^3) \times (a + b) = ?$
Ans. $a^4 - 2a^3b + 2ab^3 - b^4$.

CHAPTER IX.

DIVISION.

70. Division is finding a quotient which, multiplied by the divisor, will produce the dividend. Division is the inverse of multiplication.

In accordance with this definition and the Rule in § 66, the sign of the quotient must be + when the divisor and the dividend have like signs; and − when the divisor and the dividend have unlike signs; that is, in division, as in multiplication, we have for the signs the following

Rule.

Like signs give +; *unlike,* −.

DIVISION OF ALGEBRAIC LITERAL EXPRESSIONS.

Case I.

71. When the Divisor and Dividend are both Monomials.

1. Divide $8ax$ by $2x$.

$$8ax \div 2x = 4a$$

The coefficient of the quotient must be a number which, multiplied by 2, the coefficient of the divisor, will give 8, the coefficient of the dividend, that is, 4; and the literal part of the quotient must be a number which, multiplied by x, will give ax, that is, a; the quotient required, therefore, is $4a$.

2. Divide a^7 by a^3.

$$a^7 \div a^3 = a^4, \text{ or } \frac{aaaaaaa}{aaa} = aaaa = a^4$$

For (§ 67), $a^4 \times a^3 = a\,a\,a\,a \times a\,a\,a = a^7$. Therefore the quotient of two powers of the same number is *that number with an index equal to the index of the dividend minus the index of the divisor.*

Hence, for the division of monomials we have the following

Rule.

Annex the quotient of the letters to the quotient of their coefficients, remembering that like signs give + and unlike, —.

(3)

$$\frac{14\,a^3\,b^4\,c^2}{7\,a\,b\,c} = 2\,a^2\,b^3\,c$$

(4)

$$\frac{9\,x^2\,y^4\,z^5}{3\,x\,y\,z^2} = 3\,x\,y^3\,z^3$$

(5)

$$\frac{-\,15\,a^3\,x^2}{3\,a^2\,x} = -\,5\,a\,x$$

(6)

$$\frac{-\,10\,a^5\,c^3}{-\;5\,a^2\,c^2} = 2\,a^3\,c$$

7. $a^3\,b^2\,c \div (a^2\,b^2) = ?$
8. $a^2\,b\,c \div (-\,a\,b\,c) = ?$
9. $-\,a^3\,x^4 \div (-\,a^2\,x) = ?$
10. $35a^2b\,c^5 \div (-5\,a\,c^4) = ?$
11. $16\,b^2x^2\,y \div (-\,2\,x\,y) = ?$
12. $4\,a^2\,b^2\,c^3 \div (a\,b^2\,c^2) = ?$
13. $3\,a \div 3 = ?$
14. $a^{5m} \div a^{2m} = ?$ Ans. a^{3m}.
15. $a^m \div a^2 = ?$
16. $a^m \div a^n = ?$
17. $a^m \div a = ?$
18. $a^{p+q} \div a^q = ?$
19. $a^{p+q} \div a^p = ?$
20. $6\,a^m\,b^n \div (3\,a\,b^2) = ?$
21. $(a+b)^5 \div (a+b)^2 = ?$ Ans. $(a+b)^3$.
22. $9\,(a+b)^5 \div \{3(a+b)^2\} = ?$
23. $-\,12\,(x+y)^4 \div \{4(x+y)^2\} = ?$
24. $-\,10\,(x+y)^3 \div \{-\,2\,(x+y)^2\} = ?$
25. $18\,(x-z)^5 \div \{-\,6\,(x-z)^2\} = ?$
26. $27\,(a+2)^6 \div \{9\,(a+2)\} = ?$
27. $33\,(1-x)^7 \div \{-\,11\,(1-x)^3\} = ?$

CASE II.

72. When the Divisor only is a Monomial.

1. Divide $ax + ay + az$ by a.

From § 68, $(ax + ay + az) \div a = x + y + z$. That is, the quotient obtained by dividing the sum of two or more monomials by another is the sum of the quotients obtained by dividing the monomials separately by this other. Hence, the following

Rule.

Divide each term of the dividend by the divisor, and connect the several results by their proper signs.

(2.)

$$3a^2)\underline{15a^2b^3 - 9a^3b^4}$$
$$5b^3 - 3ab^4$$

(3.)

$$-4x)\underline{-12x^3y^5 - 8x^5y^2}$$
$$3x^2y^5 + 2x^4y^2$$

4. $(a^2 + ab) \div a = ?$

5. $(6a^3 - 3a^2b) \div (3a) = ?$

6. $(x^6 - 7x^5 + 4x^4) \div x^2 = ?$

7. $(15x^5 - 25x^4) \div (-5x^3) = ?$

8. $(-24x^6 - 32x^4) \div (-8x^3) = ?$

9. $(a^4 + a^3b + a^2b^2) \div a^2 = ?$

10. $(a^4 - a^3b + a^2b^2) \div a^2 = ?$

11. $(x^5 - x^4 + x^3 - x^2) \div x^2 = ?$

12. $(a^2bc + ab^2c + abc^2) \div (abc) = ?$

13. $(-a^3z^3 - a^2z^2 - az) \div (-az) = ?$

14. $(-34x^3 + 51x^2 - 17ax) \div (17x) = ?$

15. $(6x^4y^2 - 8x^3y^3 + 12x^2y^4) \div (2xy) = ?$

16. $(6x^4y^2 - 8x^3y^3 + 12x^2y^4) \div (-2x^2y^2) = ?$

17. $(b^2x^2z^2 - b^3x^3z^3 + b^4x^4z^4) \div (-b^2x^2) = ?$

CASE III.

73. When the Divisor and Dividend are both Polynomials.

1. Divide $a^3 - 3a^2b + 3ab^2 - b^3$ by $a^2 - 2ab + b^2$.

$$
\begin{array}{r|l}
a^2 - 2ab + b^2) & a^3 - 3a^2b + 3ab^2 - b^3 \; (a - b \\
& \underline{a^3 - 2a^2b + \;\; ab^2 \qquad} \\
& - \;\; a^2b + 2ab^2 - b^3 \\
& \underline{- \;\; a^2b + 2ab^2 - b^3}
\end{array}
$$

The divisor and dividend are arranged in the order of the powers of a, beginning with the highest power. a^3, the highest power of a in the dividend, must be the product of the highest power of a in the quotient and a^2 in the divisor; therefore $\frac{a^3}{a^2} = a$ must be the highest power of a in the quotient. The divisor, $a^2 - 2ab + b^2$, multiplied by a, must give several of the partial products which would be produced were the divisor multiplied by the whole quotient. When $(a^2 - 2ab + b^2)a = a^3 - 2a^2b + ab^2$ is subtracted from the dividend, the remainder must be the product of the divisor and the remaining terms of the quotient; therefore we treat the remainder as a new dividend, and so continue until the dividend is exhausted.

Hence, for the division of polynomials, we have the following

Rule.

Arrange the divisor and dividend in the order of the powers of one of the letters.

Divide the first term of the dividend by the first term of the divisor; the result will be the first term of the quotient.

Multiply the whole divisor by this quotient, and subtract the product from the dividend.

Consider the remainder as a new dividend, and proceed as before until the dividend is exhausted.

2. Divide $x^2 + 11x + 30$ by $x + 5$.

$$
\begin{array}{r}
x+5)\,x^2 + 11x + 30\,(x + 6 \\
\underline{x^2 + 5x} \\
6x + 30 \\
\underline{6x + 30}
\end{array}
$$

3. Divide $8a^3 + 8a^2b + 4ab^2 + b^3$ by $2a + b$.

$$
\begin{array}{l|l}
8a^3 + 8a^2b + 4ab^2 + b^3 & 2a + b \\
\cline{2-2}
\underline{8a^3 + 4a^2b} & 4a^2 + 2ab + b^2 \\
\qquad 4a^2b + 4ab^2 & \\
\qquad \underline{4a^2b + 2ab^2} & \\
\qquad\qquad 2ab^2 + b^3 & \\
\qquad\qquad \underline{2ab^2 + b^3} &
\end{array}
$$

4. Divide $12a^4 + 10a^2 - 4 + 8a - 26a^3$ by $2a^2 + 1 - 3a$.

Arrange according to the descending powers of a.

$$
\begin{array}{l|l}
12a^4 - 26a^3 + 10a^2 + 8a - 4 & 2a^2 - 3a + 1 \\
\cline{2-2}
\underline{12a^4 - 18a^3 + 6a^2} & 6a^2 - 4a - 4 \\
\qquad - 8a^3 + 4a^2 + 8a & \\
\qquad \underline{- 8a^3 + 12a^2 - 4a} & \\
\qquad\qquad - 8a^2 + 12a - 4 & \\
\qquad\qquad \underline{- 8a^2 + 12a - 4} &
\end{array}
$$

5. Divide $a^4 - b^4$ by $a - b$.

$$
\begin{array}{l}
a - b\,)\,a^4 - b^4\,(\,a^3 + a^2b + ab^2 + b^3 \\
\qquad \underline{a^4 - a^3b} \\
\qquad\qquad a^3b - b^4 \\
\qquad\qquad \underline{a^3b - a^2b^2} \\
\qquad\qquad\qquad a^2b^2 - b^4 \\
\qquad\qquad\qquad \underline{a^2b^2 - ab^3} \\
\qquad\qquad\qquad\qquad ab^3 - b^4 \\
\qquad\qquad\qquad\qquad \underline{ab^3 - b^4}
\end{array}
$$

6. Divide $x^{2n} - 3x^n y^n + 2y^{2n}$ by $x^n - y^n$.

$$
\begin{array}{l}
x^n - y^n\,)\,x^{2n} - 3x^n y^n + 2y^{2n}\,(\,x^n - 2y^n \\
\qquad\qquad \underline{x^{2n} - x^n y^n} \\
\qquad\qquad\qquad - 2x^n y^n + 2y^{2n} \\
\qquad\qquad\qquad \underline{- 2x^n y^n + 2y^{2n}}
\end{array}
$$

7. Divide $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

$$
\begin{array}{l|l}
a^3 - 3abc + b^3 + c^3 & a + b + c \\
\cline{2-2}
\underline{a^3 + a^2 b + a^2 c} & a^2 - ab - ac + b^2 - bc + c^2 \\
\quad - a^2 b - a^2 c - 3abc & \\
\quad \underline{- a^2 b - ab^2 - abc} & \\
\qquad - a^2 c + ab^2 - 2abc & \\
\qquad \underline{- a^2 c \qquad - abc - ac^2} & \\
\qquad\qquad ab^2 - abc + ac^2 + b^3 & \\
\qquad\qquad \underline{ab^2 \qquad\qquad + b^3 + b^2 c} & \\
\qquad\qquad\quad - abc + ac^2 - b^2 c & \\
\qquad\qquad\quad \underline{- abc \qquad - b^2 c - bc^2} & \\
\qquad\qquad\qquad\quad ac^2 + bc^2 + c^3 & \\
\qquad\qquad\qquad\quad \underline{ac^2 + bc^2 + c^3} &
\end{array}
$$

In the examples the dividend, divisor, and successive remainders are arranged in *descending* powers of a and x. The *ascending* powers would have answered as well. The choice of letter and kind of arrangement are immaterial, but it is especially important before beginning the division that some arrangement should be adopted and maintained throughout the operation.

8. $(x^2 - 17x + 72) \div (x - 9) = ?$ Ans. $x - 8$.

9. $(9x^2 - 3x - 2) \div (3x - 2) = ?$

10. $(6x^2 - 5x - 6) \div (2x - 3) = ?$ Ans. $3x + 2$.

11. $(9x^3 - 18x^2 + 26x - 24) \div (3x - 4) = ?$
Ans. $3x^2 - 2x + 6$.

12. $(x^3 + 27) \div (x + 3) = ?$

13. $(x^3 - 27) \div (x^2 + 3x + 9) = ?$ Ans. $x - 3$.

14. $(x^3 - y^3) \div (x - y) = ?$

15. $(x^3 + y^3) \div (x + y) = ?$

16. $(x^6 - y^6) \div (x^4 + x^2 y^2 + y^4) = ?$

17. $(x^4 + x^2 + 1) \div (x^2 + x + 1) = ?$

18. $(x^6 - y^6) \div (x^2 + xy + y^2) = ?$
Ans. $x^4 - x^3 y + x y^3 - y^4$.

19. $(x^3 - x^2 y + x y^2 - y^3) \div (x - y) = ?$

20. $(6x^4 + 23x^3 + 42x^2 + 41x + 20) \div (3x^2 + 4x + 5) = ?$
Ans. $2x^2 + 5x + 4$.

21. $(6x^4 + 32x^2 - 23x^3 - 18 - 9x) \div (6 - 5x + 2x^2) = ?$

22. $(72x^2 + 20x^4 - 33x^3 - 35x + 30) \div (4x^2 - 5x + 10) = ?$
Ans. $5x^2 - 2x + 3$.

23. $(2y^2 - 16y^3 + 2y^4 + 92y + 48) \div (-5y - 12 + y^2) = ?$

24. $(80a - 23a^3 + 3a^4 - 5a^2 + 50) \div (a^2 - 6a - 10) = ?$
Ans. $3a^2 - 5a - 5$.

25. $(x^4 - 8x^3 y + 21x^2 y^2 - 16xy^3 - 7y^4) \div (x^2 - 5xy + 7y^2) = ?$

26. $(x^4 - 9ax^3 + 12a^2x^2 + 35a^3x + 15a^4) \div (x^2 - 4ax - 3a^2) = ?$
Ans. $x^2 - 5ax - 5a^2$.

27. $(40ab^3 + 25b^4 - 4a^2b^2 + 4a^4 - 16a^3b) \div (2a^2 - 4ab - 5b^2) = ?$

28. $(4y^4 - 15by^3 + 26b^2y^2 - 23b^3y + 8b^4) \div (4y^2 - 7by + 8b^2) = ?$
Ans. $y^2 - 2by + b^2$.

29. $(31x^2y^2 + 5x^4 + 12y^4 - 14x^3y - 22xy^3)$
$\div(-4xy + 5x^2 + 3y^2) = ?$

30. $(6a^4 + 21a^3b + 31a^2b^2 + 27ab^3 - 5b^4) \div (3a^2 + 6ab - b^2) = ?$
Ans. $2a^2 + 3ab + 5b^2$.

31. $(a^4 + b^4 + 2a^2b^2 - 2c^2d^2 - c^4 - d^4) \div (a^2 + b^2 - c^2 - d^2) = ?$

32. $(6x^4 + 5x^3 + 6x^2 - 17x + 6) \div (6x^2 - 7x + 2) = ?$

33. $(9a^2b^3 - 12a^4b + 3b^5 + 2a^3b^2 + 4a^5 - 11ab^4)$
$\div (3b^3 + 4a^3 - 2ab^2) = ?$

34. $(a^6 - 3a^4b^2 + 3a^2b^4 - b^6) \div (a^3 + 3a^2b + 3ab^2 + b^3) = ?$

CHAPTER X.

THEOREMS OF DEVELOPMENT.

74. LET a and b represent any two numbers. Their sum is $a + b$; and $(a + b)^2 = (a + b)(a + b)$, which expanded becomes $a^2 + 2ab + b^2$, as appears from the following process:

$$
\begin{array}{l}
a + b \\
a + b \\
\hline
a^2 + ab \\
\quad + ab + b^2 \\
\hline
a^2 + 2ab + b^2
\end{array}
$$

From this we deduce the following

THEOREM.

The square of the sum of two numbers is equal to the square of the first, plus twice the product of the two, plus the square of the second.

According to this theorem find the square of

1. $x + y$.
2. $x + 1$.
3. $x + 2$.
4. $3x + 1$.
5. $ax + 1$.
6. $2x + a$.
7. $a^2 + b^2$.
8. $3x + 5y$.
9. $2x + 3y$.
10. $x^3 + y^3$.
11. $3x^2 + y$.
12. $2a + 5b$.
13. $3x + 4y$.
14. $5x + y$.

75. Let a and b represent any two numbers. Their difference is $a - b$; and $(a - b)^2 = (a - b)(a - b)$, which expanded becomes $a^2 - 2ab + b^2$, as appears from the following process:

$$\begin{array}{l} a - b \\ a - b \\ \hline a^2 - ab \\ \quad - ab + b^2 \\ \hline a^2 - 2ab + b^2 \end{array}$$

From this we deduce the following

THEOREM.

The square of the difference of two numbers is equal to the square of the first, minus twice the product of the two, plus the square of the second.

According to this theorem find the square of

1. $x - a$.
2. $x - 1$.
3. $x - 2$.
4. $2x - a$.
5. $a^2 - b^2$.
6. $2x - 3y$.
7. $4x - 3y$.
8. $x^3 - y^3$.
9. $ab - xy$.
10. $2x^2 - y$.
11. $5z - 3y$.
12. $3a - 5x$.
13. $5x - 2z$.
14. $7z - y$.
15. $3ab - 2c$.
16. $2ac - b$.
17. $4x - 4y$.
18. $3z - 2y$.
19. $5a - 5b$.
20. $4x - 6y$.
21. $3a - 7b$.
22. $2z - 8y$.
23. $x - 9y$.
24. $2c - 5d$.
25. $5a - 4b$.
26. $3a^2 - b^2$.
27. $4a^2b^2 - bc$.
28. $6x^2 - 5y^2$.
29. $7a^3 - 3b^3$.
30. $9x^2 - z^2$.

76. Let a and b represent any two numbers. Their sum is $a+b$, and their difference $a-b$; and $(a+b)(a-b) = a^2 - b^2$, as appears from the following process:

$$\begin{array}{l} a + b \\ a - b \\ \hline a^2 + ab \\ \quad\; - ab - b^2 \\ \hline a^2 \qquad\quad - b^2 \end{array}$$

From this we deduce the following

THEOREM.

The product of the sum and difference of two numbers is equal to the difference of their squares.

According to this theorem multiply

1. $x+y$ by $x-y$.
2. $x+3$ by $x-3$.
3. $a+1$ by $a-1$.
4. x^2+y^2 by x^2-y^2.
5. $2a+3b$ by $2a-3b$.
6. $2a+b$ by $2a-b$.
7. $3+x$ by $3-x$.
8. $5+y$ by $5-y$.
9. $2y+7$ by $2y-7$.
10. $6a+6$ by $6a-6$.

77. This theorem suggests an easy method of squaring numbers.

For, since $a^2 = (a+b)(a-b) + b^2$,

$$49^2 = (49+1)(49-1) + 1^2 = 50 \times 48 + 1$$
$$= 100 \times 24 + 1 = 2401$$

In accordance with this principle find the square of

1. 98.

$$98^2 = (98+2)(98-2) + 2^2 = 100 \times 96 + 4 = 9604$$

2. 99.
3. 98.
4. 48.
5. 47.
6. 499.
7. 493.
8. 999.
9. 991.

78. Let $x + a$, $x + b$ represent any two binomials. Then $(x + a)(x + b) = x^2 + (a + b)x + ab$, as appears from the following process:

$$\begin{array}{l} x + a \\ \underline{x + b} \\ x^2 + ax \\ \quad + bx + ab \\ \hline x^2 + (a + b)x + ab \end{array}$$

From this we deduce the following

THEOREM.

The product of two binomials of the form, $x + a$, $x + b$, is equal to the square of the first term, plus the sum of the second terms into the first term, plus the product of the second terms.

(1.)

$$\begin{array}{l} x + 5 \\ \underline{x + 3} \\ x^2 + 5x \\ \quad + 3x + 15 \\ \hline x^2 + 8x + 15 \end{array}$$

(2.)

$$\begin{array}{l} x - 5 \\ \underline{x - 3} \\ x^2 - 5x \\ \quad - 3x + 15 \\ \hline x^2 - 8x + 15 \end{array}$$

(3.)

$$\begin{array}{l} x - 5 \\ \underline{x + 3} \\ x^2 - 5x \\ \quad + 3x - 15 \\ \hline x^2 - 2x - 15 \end{array}$$

(4.)

$$\begin{array}{l} x + 5 \\ \underline{x - 3} \\ x^2 + 5x \\ \quad - 3x - 15 \\ \hline x^2 + 2x - 15 \end{array}$$

According to this theorem multiply

5. $x + 7$ by $x + 2$.
6. $x - 7$ by $x - 2$.
7. $x + 7$ by $x - 2$.
8. $x - 7$ by $x + 2$.
9. $a - 5$ by $a + 4$.
10. $y - 3x$ by $y - 5x$.
11. $x - 4a$ by $x + 2a$.
12. $x - 1$ by $x + 5$.
13. $x + 8$ by $x - 1$.
14. $x + 4a$ by $x + 2a$.
15. $a - 3b$ by $a + 5b$.
16. $x + 6a$ by $x - 5a$.

79. MISCELLANEOUS EXAMPLES.

Find the square of

1. $x + 2y$.
2. $x - 2y$.
3. $a + 3c$.
4. $a - 3c$.
5. $2a + x$.
6. $2a - x$.
7. $3a^2 + y$.
8. $3a^2 - y$.
9. $2ax + 3y$.
10. $2ax - 3yz$.
11. $2x^2 + 3y^2$.
12. $2x^2 - 3y^2$.

Find the product of

13. $x - 10$ and $x + 10$.
14. $x^2 + 12$ and $x^2 - 12$.
15. $x - 10a$ and $x + 10a$.
16. $a^2 + 9x^2$ and $a^2 - 9x^2$.
17. $2a - 3bc$ and $2a + 3bc$.
18. $a + b + c$ and $a + b - c$.

$[a+b+c][a+b-c] = [(a+b)+c][(a+b)-c] = (a+b)^2 - c^2$ (§ 76).

19. $a + b + c$ and $a - b - c$.

$[a+b+c][a-b-c] = [a+(b+c)][a-(b+c)] = ?$

20. $a + b - c$ and $a - b + c$.
21. $a - b + c$ and $a - b - c$.
22. $a - b + c$ and $a + b + c$.
23. $a^2 + a + 1$ and $a^2 - a + 1$.
24. $a + 2b + 3c$ and $a + 2b - 3c$.
25. $a + 2b + 3c$ and $a - 2b + 3c$.
26. $a + b + c + d$ and $a + b + c - d$.
27. $x^2 + xy + y^2$ and $x - y$.
28. $x^2 - xy + y^2$ and $x + y$.
29. $x - a$, $x + a$, and $x^2 + a^2$.
30. $x - 3$, $x + 3$, and $x^2 + 9$.
31. $2x - 1$, $2x + 1$, and $4x^2 + 1$.
32. $1 - ax$, $1 + ax$, and $1 + a^2x^2$.
33. $x - a$, $x + a$, $x^2 + a^2$, and $x^4 + a^4$.
34. $x - 1$, $x + 1$, $x^2 + 1$, $x^4 + 1$, and $x^8 + 1$.

CHAPTER XI.

FACTORING.

(SEE PREFACE.)

80. AN algebraic expression which contains no terms in the fractional form is called an *integral expression.*

Thus, $a^2 - b^2$, $3xy + 2z$, are integral expressions.

81. A **Root** is one of the equal factors into which a number may be resolved.

A root is indicated by the radical sign $\sqrt{}$, the initial letter of the word *radix* (*root*). The root index is written at the top of the sign, though the index denoting the second, or square, root is generally omitted. Thus,

$\sqrt{a}$; read, the second root, or the square root, of a.

$\sqrt[3]{a}$; read, the third root, or the cube root, of a.

$\sqrt[n]{a}$; read, the nth root of a.

An expression is *rational* when none of its terms contain square roots or other roots.

82. The **Factors** of an algebraic expression are the rational and integral expressions whose product is this expression.

83. A **Prime Factor** is one that is divisible without a remainder by no rational and integral expression except $\pm$ itself and ± 1.

84. The factors of a purely algebraic monomial are apparent.

Thus, the factors of ac^2x^2yz are a, c, c, x, x, y, and z.

85. Polynomials are factored in accordance with the principles of division and the theorems of the preceding chapter.

CASE I.

86. When all the Terms have a Common Factor.

1. Find the factors of $bx + by - bz$.

$$b)\,(bx + by - bz) = b\,(x + y - z)$$

$x + y - z$

As b is a factor of each term, it must be a factor of the polynomial; and if we divide the polynomial by b, we obtain the other factor. Hence the following

Rule.

Divide the given polynomial by the common factor; take the quotient thus obtained for one of the factors, and the divisor for the other.

NOTE. The greatest monomial factor is usually sought. The two factors may often be still further resolved.

Find the factors of

2. $x^3 + x^2y + xy^2$.

   Ans. x, and $x^2 + xy + y^2$.

3. $ax + a^2$.

4. $x^3 - 3ax$.

5. $x^3 - x^2$.

6. $2a - 2a^2$.

7. $3a^2 + 6ab$.

8. $15 + 25x^2$.

9. $7 - 35x$.

10. $13x - 39y$.

11. $15z + 35y$.

12. $ab - abx$.

13. $4x^2 - 2x$.

14. $6a^3 + 2a^4 + 4a^5$.

15. $3x^4 - 3x^3y + 6x^2y^2$.

16. $7a - 7a^3 + 14a^4$.

17. $5x^5 - 10a^2x^3 - 15a^3x^3$.

18. $38a^3x^5 + 57a^4x^2$.

19. $x^9 - x^3$.

20. $x^n - x^{n+2}$.

21. $2a^n + 4a^{n+1} - 6a^{n+2}$.

22. $3x^3y - 6x^4y^2 + 12x^5y^5$.

23. $7a^4b^3 + 14a^3b^2 - 28a^2b$.

24. $4xy - 24x^2y^2 - 36x^3y^3$.

25. $5b^3 - 10b^2 + 15b$.

26. $9xy - 27x^2y^2 + 45x^3y^3$.

CASE II.

87. When one Term of a Trinomial is equal to twice the Product of the Square Roots of the other two.

1. Find the factors of $a^2 + 2ab + b^2$.

$$a^2 + 2ab + b^2 = (a + b)(a + b)$$

We resolve this into its factors at once by the converse of the principle in the Theorem in § 74.

2. Find the factors of $a^2 - 2ab + b^2$.

$$a^2 - 2ab + b^2 = (a - b)(a - b)$$

We resolve this into its factors at once by the converse of the principle in the Theorem in § 75. Hence the following

Rule.

Omitting the term that is equal to twice the product of the square roots of the other two, take for each factor the square root of each of the other two connected by the sign of the term omitted.

Find the factors of

3. $a^2 + 2ab + b^2$.
4. $a^2 + 6ab + 9b^2$.
5. $a^2 - 6ab + 9b^2$.
6. $x^2 - 10xy + 25y^2$.
7. $4x^2 + 12xy + 9y^2$.
8. $81x^2 + 4y^2 - 36xy$.
9. $25a^2b^2 - 10abc + c^2$.
10. $x^2 - 2abcx + a^2b^2c^2$.
11. $4abxy + a^2x^2 + 4b^2y^2$.
12. $1 - 4x + 4x^2$.

CASE III.

88. When a Binomial is the Difference between Two Squares.

1. Find the factors of $a^2 - b^2$.

$$a^2 - b^2 = (a + b)(a - b)$$

We resolve this into its factors at once by the converse of the principle in the Theorem in § 76. Hence the following

Rule.

Take for one of the factors the sum, and for the other the difference, of the square roots of the terms of the binomial.

Find the factors of

2. $x^2 - a^2$.
3. $x^2 - 1$.
4. $4x^2 - 1$.
5. $a^2 - 4b^2$.
6. $9x^2 - y^2$.
7. $1 - 25x^2$.
8. $49 - c^2$.
9. $9 - a^2$.
10. $x^2 - 81$.
11. $9x^2 - 16y^2$.
12. $1 - a^2b^2$.
13. $25 - 64x^2$.
14. $x^4 - 16b^2$.
15. $a^2b^2 - 9x^4$.
16. $81x^6 - 25a^2$.
17. $121a^4 - b^2$.
18. $a^{10} - x^4$.
19. $a^6 - x^2$.

89. When one or both of the squares is a polynomial, the same method is employed.

1. Find the factors of $4x^2 - (y - z)^2$.

The square root of $4x^2 = 2x$.
The square root of $(y - z)^2 = y - z$.
Their sum is $2x + (y - z) = 2x + y - z$.
Their difference is $2x - (y - z) = 2x - y + z$.
Therefore $4x^2 - (y - z)^2 = (2x + y - z)(2x - y + z)$.

Find the factors of

2. $(a + b)^2 - c^2$.
3. $(a - b)^2 - c^2$.
4. $x^2 - (y + z)^2$.
5. $x^2 - (y - z)^2$.
6. $(x + y)^2 - 4x^2$.
7. $(x + 2y)^2 - a^2$.
8. $(a + b)^2 - 4c^2$.
9. $1 - (a - b)^2$.
10. $(a - 2x)^2 - b^2$.
11. $(2x - 3a)^2 - 9c^2$.
12. $(a + b)^2 - (c + d)^2$.
13. $(a - b)^2 - (c - d)^2$.
14. $(4a + x)^2 - (b + y)^2$.
15. $(5x + y)^2 - 1$.

Resolve into factors and simplify

16. $(a + 3b)^2 - (a - 2b)^2$.

$$\begin{aligned}(a + 3b)^2 - (a - 2b)^2 &= (a + 3b + a - 2b)(a + 3b - a + 2b)\\ &= (2a + b)(5b)\end{aligned}$$

17. $(x + y)^2 - x^2$.
18. $x^2 - (y - x)^2$.
19. $(x + 3y)^2 - 4y^2$.
20. $(a - b)^2 - (a + b)^2$.
21. $(a + b)^2 - (a - b)^2$.
22. $(2a + 1)^2 - (3a - 1)^2$.
23. $(7x + 3)^2 - (5x - 4)^2$.
24. $(a + b - 3)^2 - (a - b + 3)^2$.

90. Polynomials may often be arranged in two groups with the minus sign between them, and then factored as shown above.

Find the factors of

1. $(x^2 - 2xy + y^2) - z^2$.

$$x^2 - 2xy + y^2 - z^2 = (x-y)^2 - z^2 = (x - y + z)(x - y - z)$$

2. $4b^2 - a^2 + 2ax - x^2$.

$$4b^2 - a^2 + 2ax - x^2 = 4b^2 - (a^2 - 2ax + x^2)$$
$$= 4b^2 - (a-x)^2 = (2b + a - x)(2b - a + x)$$

3. $x^2 + 2xy + y^2 - a^2$.
Ans. $x + y + a$, and $x + y - a$.
4. $a^2 - 2ab + b^2 - x^2$.
5. $4a^2 + 4ab + b^2 - 9c^2$.
6. $2cd - 1 + c^2 + d^2$.
7. $x^2 - a^2 - b^2 - 2ab$.
8. $x^2 + y^2 + 2xy - 4x^2y^2$.
9. $x^4 - x^2 - 2x - 1$.
10. $1 - a^2 - 2ab - b^2$.
11. $a^4 - 1 + 2x - x^2$.
12. $a^2 + 2ab + b^2 - c^2 + 2cd - d^2$.
13. $x^2 + 2x + 1 - x^2 + 2x + 1$.
14. $(x^2 - 4ax + 4a^2) - (b^2 + 2by - y^2)$.
15. $1 - 4ab - a^2 + x^2 - 2x - 4b^2$.

91. Trinomials of the form $x^{4n} + x^{2n}y^{2n} + y^{4n}$ can be written as the difference of two squares, and factored by the above method.

1. Find the factors $a^4 + a^2b^2 + b^4$.

$$a^4 + a^2b^2 + b^4 = a^4 + 2a^2b^2 + b^4 - a^2b^2$$
$$= (a^2 + b^2)^2 - a^2b^2 = (a^2 + b^2 + ab)(a^2 + b^2 - ab)$$

Find the factors of

2. $a^8 + a^4 + 1$.

Ans. $a^2 + a + 1$, $a^2 - a + 1$, and $a^4 - a^2 + 1$.

3. $a^4 + a^2 + 1$.
4. $a^4 - 6a^2b^2 + b^4$.
5. $a^8 + a^4b^4 + b^8$.
6. $a^4 - 10a^2 + 9$.
7. $x^4 - 3x^2 + 1$.
8. $x^4 - 11x^2y^2 + y^4$.

Ans. $x^2 - y^2 + 3xy$, $x^2 - y^2 - 3xy$.

CASE IV.

92. When the Polynomials can be arranged in Groups of two or more Terms, having a Factor common to all the Groups.

1. Find the factors of $a x - a y + b x - b y$.

$$a x - a y + b x - b y = (a x - a y) + (b x - b y)$$
$$= a(x - y) + b(x - y) = (x - y)(a + b)$$

Or
$$a x - a y + b x - b y = (a x + b x) - (a y + b y)$$
$$= x(a + b) - y(a + b) = (a + b)(x - y)$$

Hence the following

Rule.

Group the terms of the expression so that each group shall have a monomial factor, then factor each group according to Case I., and finally divide by the factor common to all the groups.

This common factor, and the quotient obtained by the division, will be the factors required.

Find the factors of

2. $a c + b c + a d + b d$.
3. $a c - b c - a d + b d$.
4. $x^3 + a x^2 + a^2 x + a^3$. Ans. $x + a$, $x^2 + a^2$.
5. $x^3 - a x^2 + a^2 x - a^3$.
6. $x^3 + x^2 y - x y^2 - y^3$. Ans. $x + y$, $x + y$, $x - y$.
7. $x^3 - x^2 y - x y^2 + y^3$.
8. $x^3 + 2 x^2 + 4 x + 8$.
9. $x^3 - 3 x^2 - 9 x + 27$. Ans. $x - 3$, $x - 3$, $x + 3$.
10. $6 x^2 + 3 x y - 2 a x - a y$.
11. $a x y + b c x y - a z - b c z$. Ans. $a + b c$, $x y - z$.
12. $x - 1 + x^3 - x^2$.
13. $x^5 + a x^4 + a^2 x^3 + a^3 x^2 + a^4 x + a^5$. Ans. $x + a$, $x^2 + a x + a^2$, $x^2 - a x + a^2$.
14. $x^5 - 2 x^4 - 4 x^3 + 8 x^2 + 16 x - 32$. Ans. $x^4 - 4 x^2 + 16$, $x - 2$.
15. $a^3 + 3 a^2 b + 3 a b^2 + b^3$.
16. $a^3 - 3 a^2 b + 3 a b^2 - b^3$. Ans. $a - b$, $a - b$, $a - b$.

CASE V.

93. A Trinomial in the form, $x^2 + (a + b)\,x + a\,b$, can be separated into two Binomial Factors.

From the converse of the Theorem in § 78,

$$x^2 + (a + b)\,x + a\,b = (x + a)\,(x + b)\ldots \quad (1)$$
$$x^2 - (a + b)\,x + a\,b = (x - a)\,(x - b)\ldots \quad (2)$$
$$x^2 - (a - b)\,x - a\,b = (x - a)\,(x + b)\ldots \quad (3)$$
$$x^2 + (a - b)\,x - a\,b = (x + a)\,(x - b)\ldots \quad (4)$$
$$x^2 + 13\,x + 40 = (x + 8)\,(x + 5)\ldots \quad (5)$$
$$x^2 - 13\,x + 40 = (x - 8)\,(x - 5)\ldots \quad (6)$$
$$x^2 - 3\,x - 40 = (x - 8)\,(x + 5)\ldots \quad (7)$$
$$x^2 + 3\,x - 40 = (x + 8)\,(x - 5)\ldots \quad (8)$$

By inspecting the above results, we find that, when a trinomial is in this form,

1. The first term of *both* factors, in the trinomial, is the square root of the first term of the trinomial.

2. The second terms of the factors are such numbers that their product always equals the last term, and their sum the coefficient of the second term.

Hence, for factoring a trinomial of the form $x^2 + (a+b)\,x + a\,b$, we have the following

Rule.

Find two numbers such that their product shall equal the last term of the trinomial, and their sum the coefficient of the second term; join each number, respectively, with its proper sign, to the square root of the first term for the factors required.

1. Find the factors of $x^2 + 7\,x + 10$.

The second terms of the factors must be such that their product is $+ 10$, and their sum $+ 7$. The only pairs of integral numbers that multiplied together make $+ 10$ are ± 10 and ± 1, ± 5 and ± 2. From these we are to select that pair whose sum is $+ 7$. These are 5 and 2.

The first term of both factors is the square root of x^2.

$$\therefore\ x^2 + 7\,x + 10 = (x + 5)\,(x + 2)$$

2. Find the factors of $x^2 + x - 12$.

The only pairs of integral numbers that, multiplied together, make -12 are ± 12 and ∓ 1, ± 6 and ∓ 2, ± 4 and ∓ 3. The pair whose sum is $+1$ is $+4$ and -3. The square root of x^2 is x.

$$\therefore\ x^2 + x - 12 = (x + 4)(x - 3).$$

Find the factors of

3. $x^2 + 5x + 6$.
4. $x^2 + 7x + 12$.
5. $x^2 + 6x + 5$.
6. $x^2 + 10x + 21$.
7. $x^2 + 2x - 3$.
8. $x^2 - 2x - 3$.
9. $x^2 - 4x - 12$.
10. $x^2 - 5x + 6$.
11. $x^2 - 8x + 12$.
12. $x^2 - 7x + 6$.
13. $x^2 - 4x - 5$.
14. $x^2 + 4x - 5$.
15. $x^2 - x - 20$.
16. $x^2 + x - 20$.
17. $9x - 10 + x^2$.
18. $x^2 - 35 + 2x$.
19. $x^2 + 7ax + 10a^2$.
20. $x^2 + 12ax + 11a^2$.
21. $2x^2 - x - 1$. Write it $-(1 + x - 2x^2)$.
22. $2 + x - x^2$.
23. $110 - x - x^2$.
24. $x^2 + ax - 42a^2$.
25. $x^2 + ax - 6a^2$.
26. $a^2b^2 - 3abc - 10c^2$.
27. $x^2y^2 + xyz - 12z^2$.
28. $ax^2 - 11ax + 30a$.
29. $x^3 + 9x^2 + 14x$.

94. In the examples just given the coefficient of the highest power is unity; but when the coefficient of the highest power is not unity, we can often find by inspection the binomial factors of a trinomial.

1. Find the factors of $3x^2 + 7x + 2$.

It is evident that the first term of one of the binomial factors is $3x$, and of the other x; and that the second term of one must be 2, and of the other 1. Since the signs of $7x$ and of 2 are both +, the signs of the last terms of both binomial factors must be +. The factors must therefore be either $3x + 2$ and $x + 1$, or $3x + 1$ and $x + 2$. By trial we find that it is the second set that gives for the middle term $+7x$. Hence the factors are $3x + 1$ and $x + 2$.

2. Find the factors of $2x^2 - 5x + 3$.

The first terms of the binomial factors are $2x$ and x, and since the last term of the trinomial is + and the middle term is −, the last terms of the binomial factors are both −. Hence the factors must be either $2x - 1$ and $x - 3$, or $2x - 3$ and $x - 1$. By trial we find $2x - 3$ and $x - 1$ are the factors.

3. Find the factors of $3x^2 - 7x - 6$.

The first terms of the binomial factors are $3x$ and x, and the last terms (disregarding the signs) are either 6 and 1, or 3 and 2. As the sign of the last term of the trinomial is −, the signs of the last terms of the binomials are different. Hence the factors of the trinomial are $3x - 6$ and $x + 1$, or $3x + 1$ and $x - 6$, or $3x - 3$ and $x + 2$, or $3x + 2$ and $x - 3$. It is only the last set that will give $-7x$ for the middle term. Hence $3x + 2$ and $x - 3$ are the factors sought.

If the 3's were in the same binomial factor, the middle term would be divisible by 3. Since then the 3's cannot be in the same binomial factor, it can be seen that neither $3x - 6$ nor $3x - 3$ can be factors. This at once rules out two of the four sets named above.

Find the factors of

4. $10x^2 - 7x - 1$.
   Ans. $5x - 1$ and $2x - 1$.
5. $3x^2 + 2x - 8$.
   Ans. $x + 2$ and $3x - 4$.
6. $3x^2 + x - 2$.
7. $2x^2 + 3x + 1$.
   Ans. $2x + 1$ and $x + 1$.
8. $2x^2 + 5x + 2$.
9. $3x^2 + 10x + 3$.
10. $2x^2 + 3x - 2$.
11. $3x^2 - 13x - 10$.
12. $5x^2 - 22x + 21$.
13. $5x^2 - 32x - 21$.
14. $5x^2 + 38x + 21$.
15. $3x^2 - 5x + 2$.
16. $3x^2 - 7x + 2$.
17. $2x^2 + x - 1$.
18. $2x^2 - 3x + 1$.
19. $5x^2 - 3x - 2$.
20. $5x^2 + 9x - 2$.
21. $3x^2 + 2x - 1$.
22. $7x^2 + 4x - 3$.
23. $7x^2 - 20x - 3$.
24. $7x^2 + 20x - 3$.
25. $7x^2 - 10x + 3$.
26. $7x^2 + 10x + 3$.
27. $7x^2 - 9x + 2$.
28. $7x^2 - 5x - 2$.

Case VI.

95. When the Expression is a Binomial of the Form, $a^n \pm b^n$, n being a Positive Integer.

By actual division:

(1) $$\frac{a^2-b^2}{a-b}=a+b$$

$$\frac{a^3-b^3}{a-b}=a^2+ab+b^2$$

$$\frac{a^4-b^4}{a-b}=a^3+a^2b+ab^2+b^3$$

(2) $$\frac{a^2-b^2}{a+b}=a-b$$

$$\frac{a^4-b^4}{a+b}=a^3-a^2b+ab^2-b^3$$

(3) $$\frac{a^3+b^3}{a+b}=a^2-ab+b^2$$

$$\frac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4$$

96. By trial we shall find that

(1) $a+b$ is a factor of a^n+b^n when n is odd, but not when n is even.
(2) $a+b$ is a factor of a^n-b^n when n is even, but not when n is odd.
(3) $a-b$ is a factor of a^n-b^n always.
(4) $a-b$ is a factor of a^n+b^n never.

Note. One can always answer any one of these four points by considering the matter thus:

(1) Will $a+b$ divide $a+b$? Yes. Will $a+b$ divide a^2+b^2? No.
(2) Will $a+b$ divide $a-b$? No. Will $a+b$ divide a^2-b^2? Yes.
(3) Will $a-b$ divide $a-b$? Yes. Will $a-b$ divide a^2-b^2? Yes.
(4) Will $a-b$ divide $a+b$? No. Will $a-b$ divide a^2+b^2? No.

Attention to the following laws will enable one readily to write out the factors of such expressions.

1. The terms of the quotient are *all positive* when the divisor is $a-b$, and *alternately positive* and *negative* when the divisor is $a+b$.

2. The number of terms corresponds to the degree of the binomial.

3. a appears in the first term, b in the last term, and a and b in all the intermediate terms.

4. The exponent of a in the first term is one less than the degree of the binomial, and decreases regularly by unity in each successive term; the exponent of b in the second term is 1, and increases regularly by one in each successive term, till in the last term it becomes the same as the exponent of a in the first term.

5. The sum of the exponents of a and b in *any* term is always the same, and is equal to the exponent of a in the first term. a and b stand for any letters or expressions.

Find the factors of

1. $x^3 + y^3$. $\quad x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
2. $x^3 - y^3$. $\quad x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
3. $x^5 + y^5$. $\quad x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$
4. $c^3 - 8$. $\quad c^3 - 8 = c^3 - 2^3 = (c - 2)(c^2 + 2c + 2^2)$

5. $a^3 - b^3$.	14. $x^5 - 32$.	23. $125a^3 - 1$.
6. $x^3 + y^3$.	15. $x^3 + 64$.	24. $8a^3 - c^3d^3$.
7. $c^3 - 1$.	16. $x^3 - 64$.	25. $x^6 - y^3$.
8. $c^3 + 1$.	17. $6x^3 - 48$.	26. $27y^3 - x^6$.
9. $x^3 + 8$.	18. $27a^3 - c^3$.	27. $a^3b^3c^3 - d^3$.
10. $1 - x^3$.	19. $a^3 - 343$.	28. $a^3b^3c^3 - 1$.
11. $x^3 + 27$.	20. $a^5 - 243$.	29. $54a^3 - 2$.
12. $x^3 - 27$.	21. $125 - a^3$.	30. $24a^3 - 81b^3$.
13. $x^5 - a^5$.	22. $a^3 - 125$.	31. $32x^4 - 108xy^3$.

97. When in $a^n - b^n$ n is *even* and greater than 2, there will be three or more factors in each case, and they can be more expeditiously determined by Case III., with other principles already explained.

Find the factors of

1. $a^4 - b^4$.

$$a^4 - b^4 = (a^2 + b^2)(a^2 - b^2) = (a^2 + b^2)(a + b)(a - b)$$

2. $a^6 - b^6$.

$$a^6 - b^6 = (a^3 + b^3)(a^3 - b^3)$$
$$= (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2)$$

3. $1 - a^8$.

$$1 - a^8 = (1 + a^4)(1 - a^4) = (1 + a^4)(1 + a^2)(1 - a^2)$$
$$= (1 + a^4)(1 + a^2)(1 + a)(1 - a)$$

4. $a^8 - b^8$.	11. $a^8 - 1$.	18. $a^{13} - a^7$.
5. $x^6 - y^6$.	12. $a^{10} - b^{10}$.	19. $a^{11} - a^8$.
6. $a^4 - 1$.	13. $a^4 - 16$.	20. $a^6 - 64$.
7. $1 - a^4$.	14. $a^4 - 81$.	21. $16x^5 - x$.
8. $a^6 - 1$.	15. $81 - a^4$.	22. $a^{12} - a^6$.
9. $1 - a^6$.	16. $a^4 b^4 - c^4$.	23. $a^{12} b^{12} - a^4 b^4$.
10. $1 - a^8$.	17. $x^{12} - y^{12}$.	24. $x^{10} y^{10} - x^4 y^4$.

98. Though $a^n + b^n$ is not divisible by $a + b$ when n is *even,* it is possible to find a binomial factor in every case except when n is a power of 2, as 2, 4, 8, 16, etc.

Find the factors of

1. $x^6 + y^6$.

$$x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)[(x^2)^2 - x^2y^2 + (y^2)^2]$$
$$= (x^2 + y^2)(x^4 - x^2y^2 + y^4)$$

2. $x^{12} + y^{12}$.	**5.** $x^6 + 1$.
3. $x^{10} + y^{10}$.	6. $1 + x^{12}$.
4. $1 + x^6$.	7. $x^6 y^6 + z^6$.

99. To be expert in factoring, it is necessary to become familiar with the following algebraic expressions:

$a^2 + b^2$ is prime.
$a^2 - b^2 = (a + b)(a - b)$.
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.
$a^4 + b^4$ is prime.
$a^4 - b^4 = (a^2 + b^2)(a + b)(a - b)$.
$a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$.
$a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$.

$a^6 + b^6 = (a^2 + b^2)(a^4 - a^2 b^2 + b^4).$

$a^6 - b^6 = (a + b)(a^2 - a b + b^2)(a - b)(a^2 + a b + b^2).$

$a^7 + b^7 = (a + b)(a^6 - a^5 b + a^4 b^2 - a^3 b^3 + a^2 b^4 - a b^5 + b^6).$

$a^7 - b^7 = (a - b)(a^6 + a^5 b + a^4 b^2 + a^3 b^3 + a^2 b^4 + a b^5 + b^6).$

$a^8 + b^8$ is prime.

$a^8 - b^8 = (a^4 + b^4)(a^2 + b^2)(a + b)(a - b).$

$a^2 + 2 a b + b^2 = (a + b)^2.$

$a^2 - 2 a b + b^2 = (a - b)^2.$

$a^4 + 2 a^2 b^2 + b^4 = (a^2 + b^2)^2.$

$a^4 - 2 a^2 b^2 + b^4 = (a^2 - b^2)^2 = (a + b)^2 (a - b)^2.$

$a^6 + 2 a^3 b^3 + b^6 = (a^3 + b^3)^2 = (a + b)^2 (a^2 - a b + b^2)^2.$

$a^6 - 2 a^3 b^3 + b^6 = (a^3 - b^3)^2 = (a - b)^2 (a^2 + a b + b^2)^2.$

100. Division by Factors.

Divide

1. $a^2 - b^2$ by $a - b$.
2. $a^3 - b^3$ by $a - b$.
3. $a^3 + b^3$ by $a + b$.
4. $a^5 + b^5$ by $a + b$.
5. $a^5 - b^5$ by $a - b$.
6. $a - b$ by $b - a$.
7. $a^2 - 1$ by $a + 1$.
8. $a^3 - 1$ by $a - 1$.
9. $a^3 + 1$ by $a + 1$.
10. $1 - a^3$ by $1 - a$.
11. $4 x^2 - 9 y^2$ by $2 x - 3 y$.
12. $1 - 9 x^2$ by $1 - 3 x$.
13. $16 x^4 - y^4$ by $4 x^2 - y^2$.
14. $125 + a^3$ by $5 + a$.
15. $a^3 - b^3$ by $a^2 + a b + b^2$.
16. $a^3 + b^3$ by $a^2 - a b + b^2$.
17. $9 - x^2$ by $3 - x$.
18. $x^6 - y^6$ by $x^4 + x^2 y^2 + y^4$.
19. $27 + x^3$ by $3 + x$.
20. $x^6 - y^6$ by $x^3 + y^3$.
21. $x^2 + 7 x + 12$ by $x + 4$.
22. $x^2 - 7 x + 12$ by $x - 4$.
23. $a^2 + 2 a b + b^2$ by $a + b$.
24. $a^2 + 2 a + 1$ by $a + 1$.
25. $a^2 - 2 a b + b^2$ by $a - b$.
26. $a^2 - 2 a + 1$ by $a - 1$.
27. $x^2 - 6 x + 5$ by $x - 5$.
28. $x^2 + 9 x - 10$ by $x - 1$.
29. $x^2 - 6 a x + 9 a^2$ by $x - 3 a$.
30. $x^2 + 5 x y - 36 y^2$ by $x - 4 y$.

31. $(x+1)^2(x+2)$ by $x+1$.
32. $(x-1)(x^2-9)$ by $x-3$.
33. $(a^2-b^2)(a-b)$ by $a^2-2ab+b^2$.
34. $(a+b)^2-c^2$ by $a+b+c$.
35. $a^3-3a^2b+3ab^2-b^3$ by $a^2-2ab+b^2$.
36. $a^4+a^2b^2+b^4$ by a^2+ab+b^2.
37. $a^2+b^2+c^2+2ab+2ac+2bc$ by $a+b+c$.
38. $a^{12}+b^{12}$ by a^4+b^4.
39. $a^2+ab+ac+bc$ by $a+b$.
40. $a^2-ac+ab-bc$ by $a-c$.

101. MISCELLANEOUS EXAMPLES.

Factor the following:

1. x^4-x^2-42.
2. $x^2-9xy-10y^2$.
3. B $x^{11}-x^5$.
4. $6x+1+9x^2$.
5. $x^2-8xy+7y^2$.
6. B $x^{13}-x^5$.
7. $a^2x+ax-a-a^2$.
8. x^3+2x^2+4x+8.
9. $x^2-a^2+y^2-2xy$.
10. a^2-1+b^2-2ab.
11. B $16x^4-1$.
12. B $2a^7-2ab^6$.
13. $7x^2-50x+7$.
14. $2ax^3-22ax^2+60ax$.
15. $2a^2-a-15$.
16. $a^2+5ax+6x^2$.
17. $x^4y-x^2y^3-x^3y^2+xy^4$.
18. x^2-y^2+x-y.
19. $2bc+a^2+2ad-c^2+d^2-b^2$.
20. $a^2+b^2+c^2+2ab-2ac-2bc$.
21. $(a+b)^2+a+b$.
22. $x^8+x^4y^4+y^8$.
23. x^2-4y^2+x+2y.
24. $2xy+4-x^2-y^2$.
25. x^2-x^4-2x+1.
26. $2x^2+x-1$.
27. a^5+a^3+a.
28. a^3+3a^2+3a+1.
29. ? $a^5-8a^2b^3$.
30. $a^3x^2-c^3x^2-a^3y^2+c^3y^2$.

CHAPTER XII.

GREATEST COMMON DIVISOR.

102. A **Common Divisor** of two or more algebraic expressions is an expression which will divide each of them without remainder.

103. The **Greatest Common Divisor** of two or more algebraic expressions is the expression of the *highest degree* which will divide each of them without remainder.

104. From this definition it is evident that the greatest common divisor of two or more algebraic expressions must contain all the factors common to the expressions, and no others; and a factor may be introduced into, or rejected from, one of two algebraic expressions, *if it contains no factor of the other*, without affecting the greatest common divisor.

NOTE. The names *greatest common measure, highest common measure, highest common divisor, highest common factor*, are used by different authors to mean the same thing as *greatest common divisor*.

CASE I.

105. To find the Greatest Common Divisor of Monomials and Polynomials which can be resolved into Factors by Inspection.

1. Find the greatest common divisor of $20\,x^2y^3z^4$, $24\,x^3y$, and $28\,x^4y^2z^3$.

$$\begin{aligned} 20\,x^2y^3z^4 &= 2^2 \cdot 5\,x^2y^3z^4 \\ 24\,x^3y \quad &= 2^3 \cdot 3\,x^3y \\ \underline{28\,x^4y^2z^3} &= \underline{2^2 \cdot 7\,x^4y^2z^3} \\ \therefore\ \text{G. C. D.} &= 2^2 \cdot x^2y \end{aligned}$$

It is evident that the highest power of 2 which will divide all three expressions is 2^2; of x, x^2; of y, y; and that z will not divide them all; therefore the greatest common divisor is 2^2x^2y.

2. Find the greatest common divisor of $a^3 + b^3$, $a^2 - b^2$, and $a^2 + ab$.

$$
\begin{aligned}
a^3 + b^3 &= (a + b)(a^2 - ab + b^2) \\
a^2 - b^2 &= (a + b)(a - b) \\
a^2 + ab &= a(a + b) \\
\hline
\therefore \text{G. C. D.} &= a + b
\end{aligned}
$$

From these examples we derive the following

Rule.

Separate each expression into its prime factors; then take every factor common to the given expressions the least number of times it occurs in any one of them for the greatest common divisor required.

Find the greatest common divisor of

3. $a^2 b^3$, $a^3 b$.
4. abc, bcd.
5. $2ax$, $3x^2$.
6. $6abc$, $15bcd$.
7. a^2bc, ab^2c, abc^2.
8. $4ac^3$, $8a^2c^2$, $12abc$.
9. $12(x^2 - 9)$, $8(x - 3)^2$.
10. $x^2 + 4x + 3$, $x^2 - 1$.
11. $2x^2 + 5x - 3$, $x^2 - 9$.
12. $3a^2c^6$, $27a^4c^4$, $18a^2b^2c^4$.
13. $a^2 + ax$, $a^2 - ax$.
14. $a^2 + 2ab$, $ab + 2b^2$.
15. $(x + a)^2$, $(x + a)^3$.
16. $6(a + b)^3$, $15(a + b)^4$.
17. $2(a - b)^2$, $6(a^2 - b^2)$.
18. $x^2 - 16$, $x^2 + 4x$.
19. $ax + 7a$, $x^2 + x - 42$.
20. $x^4 - y^4$, $(x - y)^2(x + y)^2$.
21. $a^3 + x^3$, $a^2 - x^2$, $a^2 + 2ax + x^2$.
22. $x^2 - y^2$, $x^2 + xy$, $x^2y + xy^2$.
23. $x^2 - x - 6$, $x^2 + 3x - 18$.
24. $x^6 - a^6$, $(x^2 + ax + a^2)(x^2 + a^2)$.
25. $x^6 + a^6$, $(x^4 - a^2x^2 + a^4)(x^2 - a^2)$.
26. $(x - y)^5$, $(x^2 - y^2)(x - y)^3$.
27. $2x^2 - 5x - 3$, $x^2 - 8x + 15$.
28. $3x^2 - 5x - 2$, $6x^2 + 5x + 1$.

Case II.

106. To find the Greatest Common Divisor of Polynomials which cannot be factored by Inspection.

Rule.

After removing every monomial factor possible from each expression, arrange the resulting expressions, according to the descending powers of some common letter, and divide the expression which is of the higher degree by the other. Continue the division until the remainder is of a lower degree than the divisor. Then make the remainder a new divisor and the divisor a new dividend; and continue the process until there is no remainder. The last divisor, together with the common monomial factors, removed at the beginning of the operation, will be the greatest common divisor.

Note. Monomial factors should be rejected when possible, and introduced only to avoid fractions (§ 104). If, after the removal of such factors at any point of the process, polynomials of the same degree appear, either may be used as the divisor, though it is better to take as the divisor the one whose first term has the smaller coefficient.

1. Find the greatest common divisor (G. C. D.) of $x^3 + x^2 - 2$, and $x^3 + 2x^2 - 3$.

$$
\begin{array}{l}
x^3 + x^2 - 2\,)\,x^3 + 2x^2 - 3\,(\,1 \\
\qquad\qquad\quad \underline{x^3 + x^2 - 2} \\
\qquad\qquad\qquad\quad x^2 - 1\,)\,x^3 + x^2 - 2\,(\,x + 1 \\
\qquad\qquad\qquad\qquad\qquad \underline{x^3 - x\phantom{{}-2}} \\
\qquad\qquad\qquad\qquad\qquad x^2 + x - 2 \\
\qquad\qquad\qquad\qquad\qquad \underline{x^2 \phantom{{}+x} - 1} \\
\therefore\ \text{G. C. D.} = x - 1 \qquad\quad x - 1\,)\,x^2 - 1\,(\,x + 1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{x^2 - x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad x - 1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{x - 1}
\end{array}
$$

2. Find the G. C. D. of $24x^4 - 2x^3 - 60x^2 - 32x$, and $18x^4 - 6x^3 - 39x^2 - 18x$.

The following arrangement saves rewriting the divisor when it becomes the dividend:

$$
\begin{array}{r|r|r|l}
 & 2x)\,24x^4 - 2x^3 - 60x^2 - 32x & 3x)\,18x^4 - 6x^3 - 39x^2 - 18x & \\
2 & 12x^3 - x^2 - 30x - 16 & 6x^3 - 2x^2 - 13x - 6 & 2x \\
 & 12x^3 - 4x^2 - 26x - 12 & 6x^3 - 8x^2 - 8x & \\
\cline{2-3}
x & 3x^2 - 4x - 4 & 6x^2 - 5x - 6 & 2 \\
 & 3x^2 + 2x & 6x^2 - 8x - 8 & \\
\cline{2-3}
-2 & -6x - 4 & 3x + 2 & \\
 & -6x - 4 & & \\
\cline{2-2}
\end{array}
$$

$\therefore$ G. C. D. $= x(3x + 2)$

3. Find the G. C. D. of $3x^3 - 13x^2 + 23x - 21$, and $6x^3 + x^2 - 44x + 21$.

$$
\begin{array}{r|r|r|l}
x & 3x^3 - 13x^2 + 23x - 21 & 6x^3 + x^2 - 44x + 21 & 2 \\
 & 3x^3 - 10x^2 + 7x & 6x^3 - 26x^2 + 46x - 42 & \\
\cline{2-3}
-1 & -3x^2 + 16x - 21 & 9)\,27x^2 - 90x + 63 & \\
 & -3x^2 + 10x - 7 & 3x^2 - 10x + 7 & x \\
\cline{2-2}
 & 2)\,6x - 14 & 3x^2 - 7x & \\
\cline{3-3}
 & 3x - 7 & -3x + 7 & -1 \\
 & & -3x + 7 & \\
\cline{3-3}
\end{array}
$$

$\therefore$ G. C. D. $= 3x - 7$

4. Find the G. C. D. of $2x^2 - 5x + 2$, and $x^3 + 4x^2 - 4x - 16$.

$$
\begin{array}{r|r|r|l}
 & & x^3 + 4x^2 - 4x - 16 & \\
 & & 2 & \\
\cline{3-3}
2x & 2x^2 - 5x + 2 & 2x^3 + 8x^2 - 8x - 32 & x \\
 & 2x^2 - 4x & 2x^3 - 5x^2 + 2x & \\
\cline{2-3}
-1 & -x + 2 & 13x^2 - 10x - 32 & \\
 & -x + 2 & 2 & \\
\cline{2-3}
 & & 26x^2 - 20x - 64 & 13 \\
 & & 26x^2 - 65x + 26 & \\
\cline{3-3}
 & & 45)\,45x - 90 & \\
 & & x - 2 & \\
\end{array}
$$

$\therefore$ G. C. D. $= x - 2$

Find the G. C. D. of

5. $4x^2 + 3x - 10,\ 4x^3 + 7x^2 - 3x - 15.$ Ans. $4x - 5.$
6. $8x^2 + 14x - 15,\ 8x^3 + 30x^2 + 13x - 30.$
7. $x^2 - 7x + 10,\ 4x^3 - 25x^2 + 20x + 25.$
8. $x^3 + x^2 + x - 3,\ x^3 + 3x^2 + 5x + 3.$
   Ans. $x^2 + 2x + 3.$
9. $x^4 - 2x^2 + 1,\ x^4 - 4x^3 + 6x^2 - 4x + 1.$
   Ans. $x^2 - 2x + 1.$
10. $x^2 - 5xy + 4y^2,\ x^4 - 5x^3y + 4xy^3.$
11. $2x^2 - 5x + 2,\ 4x^3 + 12x^2 - x - 3.$
12. $2x^2 - 5xy + 2y^2,\ 4x^3 + 12x^2y - xy^2 - 3y^3.$
    Ans. $2x - y.$
13. $x^3 + 3x^2 + 4x + 12,\ x^3 + 4x^2 + 4x + 3.$
14. $3x^3 - 13x^2 + 23x - 21,\ 6x^3 + x^2 - 44x + 21.$
    Ans. $3x - 7.$
15. $4x^6 + 12x^4y - 40x^2y^2,\ 2x^4 - 6x^2y + 4y^2.$
16. $2a^3 - 5a^2 + 2a,\ 2a^4 - 3a^3 - 8a^2 + 12a.$
    Ans. $a(a - 2).$
17. $12b^2 - 30b + 12,\ 36b^3 - 24b^2 - 9b + 6.$
    Ans. $3(2b - 1).$
18. $x^4 - 3x^3 + 2x^2,\ x^4 - 3x^2 + 2x.$
19. $2x^3 - x + x^4 - 2,\ 2x^2 + x + 3 - x^3 + x^4.$
    Ans. $x^2 + x + 1.$
20. $2xy^2 - 2y^3 + 6x^3 - 6x^2y,\ 8x^2y + 2y^3 - 10xy^2.$
    Ans. $2(x - y).$
21. $x^4 + 4x^2 + 16,\ 2x^4 - x^3 + 16x - 8.$
22. $3x^3 + 5x^2 - x + 2,\ 4x^4 + 9x^3 + 2x^2 - 2x - 4.$
    Ans. $x + 2.$
23. $2a^4 + a^3b - 4a^2b^2 - 3ab^3,\ 4a^4 + a^3b - 2a^2b^2 + ab^3.$
24. $6x^4 + x^3 - x,\ 4x^4 - 6x^3 - 4x^2 + 3x.$
25. $4x^3 - 18x^2 + 19x - 3,\ 2x^4 - 12x^3 + 19x^2 - 6x + 9.$

CHAPTER XIII.

LEAST COMMON MULTIPLE.

107. A **Common Multiple** of two or more algebraic expressions is an expression that can be divided by each of them without remainder.

108. The **Least Common Multiple** of two or more algebraic expressions is the expression of the *lowest degree* that can be be divided by each of them without remainder.

109. It is evident from the above definitions that a common multiple of two or more expressions must contain all the different factors of these expressions; and the *least* common multiple of two or more expressions must contain *only* the factors of these expressions.

Case I.

110. To find the Least Common Multiple of Monomials, and Polynomials which can be resolved into Factors by Inspection.

1. Find the least common multiple of $16a^3b^3$, $12a^5b^3c^3$, and $20a^2b^2c$.

$$\begin{aligned} 16a^3b^3 &= 2^4\cdot a^3b^3 \\ 12a^5b^3c^3 &= 2^2\cdot 3a^5b^3c^3 \\ 20a^2b^2c &= 2^2\cdot 5a^2b^2c \\ \hline \therefore \text{L. C. M.} &= 2^4\cdot 3\cdot 5a^5b^3c^3 \end{aligned}$$

It is evident that no number which contains a power of 2 less than 2^4, of a less than a^5, of b less than b^3, of c less than c^3, and which does not contain 3 and 5, can be divided by each of these numbers; therefore the least common multiple is $2^4\cdot 3\cdot 5a^5b^3c^3$.

2. Find the least common multiple (L. C. M.) of $4(x-y)^2$, $3(x^2-y^2)$, $6(x^2+2xy+y^2)$.

$$\begin{aligned} 4(x-y)^2 &= 2^2(x-y)^2 \\ 3(x^2-y^2) &= 3(x+y)(x-y) \\ 6(x^2+2xy+y^2) &= 2 \,.\, 3(x+y)^2 \\ \hline \therefore \text{ L. C. M.} &= 2^2 \cdot 3(x+y)^2(x-y)^2 \end{aligned}$$

From these examples we derive the following

Rule.

Separate each expression into its prime factors, and then take every factor the greatest number of times it occurs in any one of the expressions for the least common multiple required.

Find the least common multiple of

3. $4a^3$, $6a^9$.
4. $6x^2y$, $15xy^2$.
5. ab, ac, bc.
6. $10x^4$, $6x^2y^2$, $12x^3y$.
7. x^4, $4x^3y$, $6x^2y^2$, $4xy^3$, y^4.
8. ay, az^2, a^2z, az.
9. $3x$, $3(a-x)$.
10. abc, $ab(a-c)$.
11. $2a^2(a+x)$, $4ax$.
12. $3(a+b)$, $7(a+b)$.
13. $a^2b(x-y)$, $ab^2(x-y)$.
14. $a^3(x+a)$, a^2b^2, $b^2(x-a)$.
15. $a(x+a)$, $b(x+a)$, $c(x+a)$.
16. x^2-y^2, $(x-y)^2$.
17. $4(x-y)$, x^3-y^3.
18. x^2-x, x^2-1.
19. x^2-a^2, $(x+a)^2$, $(x-a)^2$.
20. $(x+1)^3$, $4(x^2-1)$, 6.
21. $x(x-y)$, $x(x^2-y^2)$, x.
22. $x+y$, x^2-y^2, x^2+y^2.
23. $x-3$, $x+3$, x^2-9.
24. $ax+by$, $ax-by$, $a^2x^2-b^2y^2$.
25. $(x-a)^2$, $(x-a)(x-b)$.
26. x^3-8, x^2-4x+4.
27. $(x-y)^3$, x^3-y^3.
28. x^2-y^2, x^2-xy+y^2, x^2+xy+y^2.
29. $(x+3)^3$, x^3+27.
30. x^2-y^2, x^3+y^3, x^2-xy+y^2.
31. x^2-y^2, x^3-y^3, x^2-xy+y^2.

CASE II.

111. To find the Least Common Multiple of Polynomials which cannot be readily factored by Inspection.

Rule.

Divide one of the expressions by their greatest common divisor, and multiply this quotient by the other expression for the least common multiple required.

1. Find the least common multiple (L. C. M.) of $x^3 - 4x + 15$, $x^4 + x^2 + 25$.

$$
\begin{array}{r|l|l|l}
x & x^3 - 4x + 15 & x^4 + \quad x^2 + 25 & x \\
 & \underline{x^3 - 3x^2 + 5x} & \underline{x^4 - 4x^3 + 15x} & \\
3 & 3x^2 - 9x + 15 & 5\,)\,5x^2 - 15x + 25 & \\
 & \underline{3x^2 - 9x + 15} & \quad x^2 - \ 3x + \ 5 = \text{G. C. D.} &
\end{array}
$$

$$\therefore \qquad \text{L. C. M.} = \frac{(x^3 - 4x + 15)(x^4 + x^2 + 25)}{x^2 - 3x + 5}$$

$$= (x + 3)(x^4 + x^2 + 25)$$

$$\text{or } (x^2 + 3x + 5)(x^3 - 4x + 15)$$

$$\text{or } (x + 3)(x^2 + 3x + 5)(x^2 - 3x + 5)$$

Find the L. C. M. of

2. $x^2 + 6x + 8$, $x^3 + 5x^2 + 7x + 2$.
Ans. $(x + 2)(x + 4)(x^2 + 3x + 1)$.

3. $4a^2 - 5ab + b^2$, $3a^3 - 3a^2b + ab^2 - b^3$.
Ans. $(4a - b)(a - b)(3a^2 + b^2)$.

4. $x^2 - 7x + 10$, $4x^3 - 25x^2 + 20x + 25$.
Ans. $(x - 5)(x - 2)(4x^2 - 5x - 5)$.

5. $4x^2 + 3x - 10$, $4x^3 + 7x^2 - 3x - 15$.
Ans. $(4x - 5)(x + 2)(x^2 + 3x + 3)$.

6. $x^3 - 8x + 3$, $x^6 + 3x^5 + x + 3$.
Ans. $(x + 3)(x^2 - 3x + 1)(x^5 + 1)$.

7. $x^3 - 6x^2 + 11x - 6$, $x^3 - 9x^2 + 26x - 24$.
Ans. $(x - 1)(x - 2)(x - 3)(x - 4)$.

CHAPTER XIV.

FRACTIONS.

112. WHEN division is expressed by writing the dividend over the divisor with a line between, the expression is called a **Fraction.** As a fraction, the dividend is called the numerator, and the divisor the denominator.

Thus, $\frac{a\,b}{b}$ is a fraction whose numerator is $a\,b$ and denominator b, and whose value is a.

113. From the principles of division, it follows that

$$+\frac{+\,a\,b}{+\,a}=+\frac{-\,a\,b}{-\,a}=-\frac{-\,a\,b}{+\,a}=-\frac{+\,a\,b}{-\,a}=+\,b$$

and
$$+\frac{-\,a\,b}{+\,a}=+\frac{+\,a\,b}{-\,a}=-\frac{+\,a\,b}{+\,a}=-\,b$$

That is, (a) The value of a fraction *is not changed,*

(1) *If the sign of every term of the numerator and denominator is changed.*

(2) *If the sign of every term of the numerator and the sign before the fraction are changed.*

(3) *If the sign of every term of the denominator and the sign before the fraction are changed.*

(b) But the value of a fraction *is changed,*

(4) *If every sign of the numerator, or denominator, or the sign before the fraction, is changed.*

114. EXERCISES.

Express the following fractions in four different ways :

1. $\dfrac{a-b}{c-d}=\dfrac{b-a}{d-c}=-\dfrac{b-a}{c-d}=-\dfrac{a-b}{d-c}$

2. $\dfrac{3a}{x-y}$.

3. $\dfrac{x-y}{z}$.

4. $\dfrac{5x}{a^2-b^2}$.

5. $\dfrac{x^2-1}{x^2-x-2}$.

6. $\dfrac{b-a}{y-x}$.

7. $\dfrac{3x-1}{1-x^2}$.

8. $$\frac{a-c}{(a-b)(x-a)}=\frac{c-a}{(b-a)(x-a)}=\frac{c-a}{(a-b)(a-x)}$$
$$=\frac{a-c}{(b-a)(a-x)}$$

Changing the signs of an odd number of factors of a product changes the sign of the product; but changing the signs of an even number of factors of a product does not change the sign of the product.

Write the following fractions so that x and y shall become positive :

9. $\dfrac{-x}{z-y}=\dfrac{x}{y-z}$.

10. $\dfrac{z-x}{-y}$.

11. $\dfrac{a-x-y}{(a-b)(c-d)}$.

12. $\dfrac{a-x}{b-y}$.

13. $\dfrac{a-x-y}{z^2-9}$.

14. $\dfrac{z-y}{-x(x^2-1)}$.

REDUCTION OF FRACTIONS.

115. Reduction of Fractions is changing their form without changing their value.

CASE I.

116. To reduce a Fraction to its Lowest Terms.

A fraction is in its *lowest terms* when its numerator and denominator have no common factor, that is, are mutually prime.

$$\frac{a}{b}=\frac{am}{bm}, \quad \text{or} \quad \frac{am}{bm}=\frac{a}{b}$$

That is, *multiplying or dividing both numerator and denominator by the same number does not change the value of the fraction.*

Hence, to reduce a fraction to its lowest terms, we have the following

Rule.

Divide both terms of the fraction by any factor common to them; then divide these quotients by any factor common to them; and so proceed till the terms are mutually prime. Or,

Divide both terms by their greatest common divisor.

Reduce the following fractions to their lowest terms:

1. $\dfrac{12a^2b^2x}{18a^4b^2y}$

$$\frac{12a^2b^2x}{18a^4b^2y}=\frac{6b^2x}{9a^2b^2y}=\frac{2x}{3a^2y}$$

2. $\dfrac{x^2-xy}{x^2y-xy^2}$. Ans. $\dfrac{1}{y}$.

3. $\dfrac{abc^2}{b^2c}$.

4. $\dfrac{5b^4x}{20b^2x^4}$.

5. $\dfrac{ab^4x}{3ab^2x^2}$.

6. $\dfrac{ax^2}{x^2y-x^2}$.

7. $\dfrac{ax-bx}{abx}$.

8. $\dfrac{7a^2b}{14a^3+21a^2c}$.

9. $\dfrac{a^2+ab}{ab+b^2}$.

10. $\dfrac{(a-b)^2}{a^2-b^2}$.

11. $\dfrac{a^2-b^2}{a^3+b^3}$.

12. $\dfrac{a^3 - b^3}{a^2 - b^2}$.

13. $\dfrac{x - y}{y^2 - x^2}$.

14. $\dfrac{b^2 - a^2}{(a + b)^2}$.

15. $\dfrac{x - 2}{4 - x^2}$.

16. $\dfrac{x^2 + 4xy}{x^2 - 16y^2}$.

17. $\dfrac{x^4 - 1}{x^6 - 1}$.

18. $\dfrac{a^2 - 2ax + x^2}{a^2 - x^2}$.

19. $\dfrac{2x^3 - 4x^4}{x^2 - 4x^4}$.

20. $\dfrac{3x^2 - 12ax}{48a^2 - 3x^2}$.

21. $\dfrac{a^3 + 2a^2x + ax^2}{a^4 - a^2x^2}$.

22. $\dfrac{5ax - 15a^2}{x^2 - 9a^2}$.

23. $\dfrac{x^2 - 5x + 6}{x^2 - 7x + 12}$.

24. $\dfrac{x^2 + 5xy + 6y^2}{x^2 + xy - 6y^2}$.

25. $\dfrac{x^2 + 4x + 4}{x^3 + 8}$.

26. $\dfrac{a^2 + 5a + 6}{3a^2 + 2a - 8}$.

27. $\dfrac{a^4 - b^4}{(a^3 - b^3)(a + b)}$.

28. $\dfrac{(a - b)^2 - c^2}{(a - b + c)^2}$.

29. $\dfrac{(c - a)(c - b)}{(a - b)(a - c)(b - c)}$.

30. $\dfrac{x^2 - xy + x - y}{x^2 + 2x + 1}$.

31. $\dfrac{(a + b)^2 - (c + d)^2}{(a - c)^2 - (d - b)^2}$. Ans. $\dfrac{a + b + c + d}{a - b - c + d}$.

32. $\dfrac{a^2 + ab + ac + bc}{a^2 + ab - ac - bc}$. Ans. $\dfrac{a + c}{a - c}$.

33. $\dfrac{x^2 - 10x + 21}{x^3 - 11x + 6}$.

By § 106 the G. C. D. is found to be $x - 3$.

$$\frac{(x^2 - 10x + 21) \div (x - 3)}{(x^3 - 11x + 6) \div (x - 3)} = \frac{x - 7}{x^2 + 3x - 2}, \text{ Ans.}$$

34. $\dfrac{x^3 - 8x - 3}{x^4 - 7x^2 + 1}$. Ans. $\dfrac{x - 3}{x^2 - 3x + 1}$.

35. $\dfrac{x^2 - 10x + 21}{x^3 - 46x - 21}$. Ans. $\dfrac{x - 3}{x^2 + 7x + 3}$.

36. $\dfrac{x^3-3x+2}{x^3+4x^2-5}$. Ans. $\dfrac{x^2+x-2}{x^2+5x+5}$.

37. $\dfrac{x^3+x^2+3x-5}{x^2-4x+3}$. Ans. $\dfrac{x^2+2x+5}{x-3}$.

38. $\dfrac{x^3+3x^2-20}{x^4-x^2-12}$. Ans. $\dfrac{x^2+5x+10}{x^3+2x^2+3x+6}$.

39. $\dfrac{x^3-x^2-7x+3}{x^4+2x^3+2x-1}$. Ans. $\dfrac{x-3}{x^2+1}$.

40. $\dfrac{a^3-a^2b-ab^2-2b^3}{a^3+3a^2b+3ab^2+2b^3}$. Ans. $\dfrac{a-2b}{a+2b}$.

41. $\dfrac{1+2x^2+x^3+2x^4}{1+3x^2+2x^3+3x^4}$. Ans. $\dfrac{1-x+2x^2}{1-x+3x^2}$.

Case II.

117. To reduce a Fraction to an Integral or Mixed Number.

1. Reduce $\dfrac{x^2+5x-1}{x+2}$ to an integral or mixed number.

$$\begin{array}{r|l} x+2)\ x^2+5x-1\ (& x+3+\dfrac{-7}{x+2} \\ \underline{x^2+2x\quad\ } & \quad \text{or} \\ 3x-1 & x+3-\dfrac{7}{x+2}\quad (\S\ 113) \\ \underline{3x+6} & \\ -7 & \end{array}$$

Since a fraction represents the quotient of the numerator divided by the denominator, we perform the indicated division, adding to the quotient the fraction formed by placing the remainder over the divisor. Hence,

To reduce a fraction to an integral or mixed number, we have the following

Rule.

Divide the numerator by the denominator, and if there is a remainder place it over the divisor, and add the fraction so formed to the quotient.

Reduce the following to integral or mixed numbers :

2. $\dfrac{a^2+ab+c}{a}$.

3. $\dfrac{a^2-ab+b^2}{a}$.

4. $\dfrac{4ax-8bx-c}{2x}$.

5. $\dfrac{x^2+4x-3}{x+2}$.

6. $\dfrac{a^3-b^3}{a-b}$.

7. $\dfrac{a^3-1}{a+1}$.

8. $\dfrac{1}{1-a}$.

9. $\dfrac{1+a^3}{1+a}$.

10. $\dfrac{x^4+x^3+x^2+1}{x^2+x+1}$.

11. $\dfrac{a^5-b^5}{a-b}$.

12. $\dfrac{a^4+a^2b^2+b^4}{a^2+ab+b^2}$.

13. $\dfrac{a^4+a^2+1}{a^2-1}$. Ans. $a^2+2+\dfrac{3}{a^2-1}$.

14. $\dfrac{x^2-4ax+3a^2}{x-2a}$. Ans. $x-2a-\dfrac{a^2}{x-2a}$.

15. $\dfrac{a^4-a^3+a^2+1}{a^2+a+1}$. Ans. $a^2-2a+2-\dfrac{1}{a^2+a+1}$.

CASE III.

118. To reduce a Mixed Number to a Fractional Form.

1. Reduce $a+b+\dfrac{b^2}{a-b}$ to a fractional form.

$$a+b+\frac{b^2}{a-b}=\frac{a^2-b^2+b^2}{a-b}=\frac{a^2}{a-b}$$

2. Reduce $x-y+\dfrac{x^2+y^2}{x+y}$ to a fractional form.

$$x-y+\frac{x^2+y^2}{x+y}=\frac{x^2-y^2+(x^2+y^2)}{x+y}$$

$$=\frac{x^2-y^2+x^2+y^2}{x+y}=\frac{2x^2}{x+y}$$

3. Reduce $x + y - \dfrac{x^2 + y^2}{x - y}$ to a fractional form.

$$x + y - \frac{x^2 + y^2}{x - y} = \frac{x^2 - y^2 - (x^2 + y^2)}{x - y}$$

$$= \frac{x^2 - y^2 - x^2 - y^2}{x - y} = \frac{-2y^2}{x - y}, \text{ or } \frac{2y^2}{y - x}$$

It will be observed by referring to Case II., of which this is the converse, that the integral part of the mixed expression always stands for the quotient, the denominator of the fractional part for the divisor, and the numerator for the remainder; and that the dividing line also performs the office of a vinculum for the numerator. Hence,

To reduce a mixed number to a fractional form, we have the following

Rule.

Multiply the integral part by the denominator of the fraction; to the product add the numerator if the sign of the fraction is plus, and subtract it if the sign is minus, and under the result write the denominator.

Reduce to fractional form the following examples :

4. $a + \dfrac{b}{c}$.

5. $1 + \dfrac{1}{x}$.

6. $1 - \dfrac{1}{x}$.

7. $x + \dfrac{1}{y}$.

8. $x - \dfrac{1}{y}$.

9. $\dfrac{a}{b} - c$.

10. $1 + \dfrac{1}{1 - x}$.

11. $\dfrac{1}{1 - x} - 1$.

12. $\dfrac{a + b}{a - b} - 1$.

13. $a - x + \dfrac{x^2}{a + x}$.

14. $x + y - \dfrac{2xy + y^2}{x + y}$.

15. $a + 2b + \dfrac{3b^2}{a - b}$.

16. $\dfrac{9b}{a - 3b} + a + 3b$.

17. $x + 1 - \dfrac{x^2 + 1}{x - 1}$.

18. $\frac{a^2 - b^2}{b} - (a - b)$.

19. $a^2 - 2ax + 4x^2 - \frac{6x^3}{a + 2x}$. Ans. $\frac{a^3 + 2x^3}{a + 2x}$.

20. $\frac{a^2 - ay + y^2}{x + a} + x - a + y$. Ans. $\frac{x^2 + xy + y^2}{x + a}$.

21. $1 + \frac{a^2 + b^2 - c^2}{2ab}$. Ans. $\frac{(a + b + c)(a + b - c)}{2ab}$.

22. $1 - \frac{c^2 - a^2 - 1}{2a}$. Ans. $\frac{(a + c + 1)(a - c + 1)}{2a}$.

23. $x^3 - x^2 - x - 1 + \frac{2x^3}{x - 1}$. Ans. $\frac{x^4 + 1}{x - 1}$.

24. $b^2 + 3b - 1 - \frac{3 - 10b}{b - 3}$. Ans. $\frac{b^3}{b - 3}$.

Case IV.

119. To reduce Fractions to Equivalent Fractions having their Least Common Denominator.

1. Reduce $\frac{a}{mx}$, $\frac{b}{my}$, and $\frac{c}{mz}$ to equivalent fractions having the least common denominator.

The required fractions must have for their denominators $mxyz$, the least common multiple of the given denominators.

If we multiply the numerators, a, b, c, by the quotients of $mxyz$ divided by mx, my, mz, respectively, we shall have the equivalent fractions required.

That is,

$$\frac{a}{mx} = \frac{a \times (yz)}{mx \times (yz)} = \frac{ayz}{mxyz}$$

$$\frac{b}{my} = \frac{b \times (xz)}{my \times (xz)} = \frac{bxz}{mxyz}$$

$$\frac{c}{mz} = \frac{c \times (xy)}{mz \times (xy)} = \frac{cxy}{mxyz}$$

Hence the following

Rule.

Find the least common multiple of the denominators for the least common denominator. For new numerators, multiply each numerator by the quotient arising from dividing this multiple by its denominator.

NOTE 1. Fractions should always first be reduced to their lowest terms.

NOTE 2. The familiar method of multiplying together the denominators for a new denominator, and each numerator by all the denominators except its own, for a new numerator, is sometimes useful. When the denominators are mutually prime, it is identical with the process above.

NOTE 3. Every integral form may be considered as a fraction with unity for its denominator; that is, $a = \frac{a}{1}$.

2. Reduce $\frac{a}{2xy}$, $\frac{b}{4xy^2}$, and $\frac{c}{8x^2y}$ to equivalent fractions having the least common denominator (L. C. D.).

The L. C. D. is $8x^2y^2$.

$$\frac{a}{2xy},\quad \frac{b}{4xy^2},\quad \text{and}\quad \frac{c}{8x^2y}$$

are equal, respectively, to

$$\frac{4axy}{8x^2y^2},\quad \frac{2bx}{8x^2y^2},\quad \text{and}\quad \frac{cy}{8x^2y^2}$$

3. Reduce $\frac{1}{a}$, $\frac{2}{a(a-b)}$, and $\frac{3}{b(a+b)}$ to equivalent fractions having the least common denominator.

The L. C. D. is $ab(a^2-b^2)$.

$$\frac{1}{a},\quad \frac{2}{a(a-b)},\quad \text{and}\quad \frac{3}{b(a+b)}$$

are equal, respectively, to

$$\frac{b(a^2-b^2)}{ab(a^2-b^2)},\quad \frac{2b(a+b)}{ab(a^2-b^2)},\quad \text{and}\quad \frac{3a(a-b)}{ab(a^2-b^2)}$$

Reduce to equivalent fractions having their L. C. D.:

4. $\frac{a}{4}, \frac{b}{6}.$

5. $\frac{2x}{3}, \frac{x}{6}, \frac{3x}{12}.$

6. $\frac{4a}{5b}, \frac{3a}{10c}.$

7. $\frac{a}{3xy}, \frac{b}{6xyz}, \frac{c}{2yz}.$

8. $\frac{x}{y}, \frac{y}{x}, 3x.$

9. $\frac{a}{bc}, \frac{b}{ca}, 1.$

10. $\frac{x-1}{2}, \frac{x-2}{3}, \frac{x-3}{6}.$

11. $\frac{a-b}{ab}, \frac{a-c}{ac}, \frac{b-c}{bc}.$

12. $\frac{1}{a-b}, \frac{1}{a^2-b^2}.$

13. $\frac{1}{x-2}, \frac{2}{(x-2)^2}.$

14. $\frac{1}{x-y}, \frac{2}{x^2-y^2}.$

15. $\frac{xy}{25x^2-y^2}, \frac{x}{5x+y}.$

16. $\frac{1}{x(x+y)}, \frac{1}{y(x-y)}.$

17. $\frac{y}{x(x^2-y^2)}, \frac{x}{y(x^2+y^2)}.$

18. $\frac{a}{4(a-b)}, \frac{b}{8(a^2-b^2)}.$

19. $\frac{5}{1+2x}, \frac{6}{1-2x}, \frac{2-3x}{1-4x^2}.$

20. $\frac{1}{2a(a-b)}, \frac{2}{3b(a^2-b^2)}, \frac{3}{6c(a+b)}.$

120. Addition and Subtraction of Fractions.

1. Find the sum of $\frac{a}{b}$ and $\frac{c}{d}$, and also their difference.

$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}$$

$$\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd}$$

Hence, for adding or subtracting fractions, we have the following

Rule.

Reduce the fractions, if necessary, to equivalent fractions having their least common denominator; then add or subtract their numerators, as the sign before the fraction directs, and write the result over the least common denominator.

2. Simplify $\frac{x-3}{4x}+\frac{x+3}{2x}$.

The L. C. D. is $4x$.

$$\frac{x-3}{4x}+\frac{x+3}{2x}=\frac{x-3+2(x+3)}{4x}$$

$$=\frac{x-3+2x+6}{4x}=\frac{3(x+1)}{4x}$$

3. Simplify $\frac{x-3}{4}+\frac{x-4}{3}$.

The L. C. D. is 12.

$$\frac{x-3}{4}+\frac{x-4}{3}=\frac{3(x-3)+4(x-4)}{12}$$

$$=\frac{3x-9+4x-16}{12}=\frac{7x-25}{12}$$

Simplify

4. $\frac{x}{6}+\frac{x-2}{3}$.

5. $\frac{x+3}{2}-\frac{x-6}{3}$.

6. $\frac{x-3}{6}-\frac{x-5}{4}$.

7. $\frac{1}{a}+\frac{1}{b}$.

8. $\frac{1}{ax}+\frac{1}{bx}$.

9. $\frac{a}{x}+\frac{a^2}{x^2}$.

10. $\frac{x-a}{a}-\frac{x-b}{b}$.

11. $\frac{1}{x^2}+\frac{2}{ax}+\frac{1}{a^2}$.

12. $\frac{a}{x}+\frac{a}{2x}+\frac{a}{3x}$.

13. Simplify $\frac{a-b}{a+b}+\frac{a+b}{a-b}$.

The L. C. D. is a^2-b^2. $\frac{a-b}{a+b}+\frac{a+b}{a-b}=\frac{(a-b)^2+(a+b)^2}{a^2-b^2}$

$$=\frac{a^2-2ab+b^2+a^2+2ab+b^2}{a^2-b^2}=\frac{2a^2+2b^2}{a^2-b^2}$$

Simplify

14. $\frac{1}{a+b}+\frac{1}{a-b}$.

15. $\frac{1}{x-y}-\frac{1}{x+y}$.

16. $\frac{x}{x+a}+\frac{x}{x-a}$.

17. $\frac{x}{x+a}-\frac{x}{x-a}$.

18. $\frac{2}{x-1}-\frac{1}{x}$.

19. $\frac{x+y}{4y}-\frac{x}{x+y}$.

20. $\frac{a-b}{4a}+\frac{b}{a-b}$.

21. $\frac{a}{x(a-x)}+\frac{1}{a-x}$.

22. $\frac{a}{x(a-x)}-\frac{x}{a(a-x)}$.

23. $\frac{x}{x-1}+\frac{1}{(x-1)^2}$.

24. $\frac{1}{a+b}+\frac{1}{a-b}+\frac{2b}{a^2-b^2}$.

25. $\frac{x}{a+x}+\frac{a}{a-x}+1$.

26. $\frac{a}{a-x}+\frac{ax}{x^2-a^2}$.

27. $\frac{1}{a+b}+\frac{1}{a-b}-\frac{2b}{b^2-a^2}$.

121. Multiplication and Division of Fractions.

1. Multiply $\frac{a}{b}$ by $\frac{c}{d}$.

Let $\frac{a}{b}=x$, and $\frac{c}{d}=y$

then $a=bx$, and $c=dy$

and $ac=bdxy$ (Ax. 3)

$\frac{ac}{bd}=xy$ (Ax. 4)

But $xy=\frac{a}{b}\times\frac{c}{d}$

$\therefore$ $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$

Hence, for the multiplication of fractions, we have the following

Rule.

Multiply the numerators together for a new numerator, and the denominators for a new denominator.

2. Divide $\frac{a}{b}$ by $\frac{c}{d}$.

Let $\frac{a}{b} = x$, and $\frac{c}{d} = y$

Then $a = bx$, and $c = dy$

$ad = bdx$, and $bc = bdy$ (Ax. 3)

$\therefore \quad \frac{ad}{bc} = \frac{bdx}{bdy} = \frac{x}{y}$

But $\frac{x}{y} = x \div y = \frac{a}{b} \div \frac{c}{d}$

$\therefore \quad \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$

Hence, for the division of fractions, we have the following

Rule.

Invert the divisor, and then proceed as in multiplication of a fraction by a fraction.

Note 1. As a mixed number can be reduced to a fractional form, § 119, and an integral expression can be expressed as a fraction by writing under it 1 as its denominator, these rules cover all possible cases in multiplication and division of fractions.

Note 2. *Always cancel, if possible.*

3. Multiply $\frac{6a^5}{35x^3}$ by $\frac{14x^2}{9a^3}$.

$$\frac{6a^5}{35x^3} \times \frac{14x^2}{9a^3} = \frac{3 \cdot 2 \cdot 2 \cdot 7a^5x^2}{7 \cdot 5 \cdot 3 \cdot 3a^3x^3} = \frac{4a^2}{15x}$$

4. Multiply $\frac{a-b}{a^2+ab}$ by $\frac{a^2-b^2}{a^2-ab}$.

$$\frac{a-b}{a^2+ab} \times \frac{a^2-b^2}{a^2-ab} = \frac{a-b}{a(a+b)} \times \frac{(a+b)(a-b)}{a(a-b)} = \frac{a-b}{a^2}$$

5. Divide $\frac{4a^2b}{5x^2y}$ by $\frac{2ab^2}{15xy}$.

$$\frac{4a^2b}{5x^2y} \div \frac{2ab^2}{15xy^2} = \frac{4a^2b}{5x^2y} \times \frac{15xy^2}{2ab^2} = \frac{2 \cdot 2 \cdot 3 \cdot 5a^2bxy^2}{5 \cdot 2ab^2x^2y} = \frac{6ay}{bx}$$

6. Divide $\frac{a^2 - 4x^2}{a^2 + 4ax}$ by $\frac{a^2 - 2ax}{ax + 4x^2}$.

$$\frac{a^2 - 4x^2}{a^2 + 4ax} \div \frac{a^2 - 2ax}{ax + 4x^2} = \frac{a^2 - 4x^2}{a^2 + 4ax} \times \frac{ax + 4x^2}{a^2 - 2ax}$$

$$= \frac{(a + 2x)(a - 2x)}{a(a + 4x)} \times \frac{x(a + 4x)}{a(a - 2x)} = \frac{x(a + 2x)}{a^2}$$

Simplify

7. $\frac{4x^3}{a^2} \times \frac{a}{2x^2}$.

8. $\frac{3x^2}{4z} \times \frac{5y^2}{6xy} \times \frac{12xz^2}{20x^3y^2}$.

9. $\frac{4x^3}{a^2} \div \frac{2a}{x}$.

10. $\frac{6a^5}{14x^3} \div \frac{9a^3}{35x^2}$.

11. $\frac{x + 3}{x(x - 3)} \times \frac{x^2}{x + 3}$.

12. $\frac{x^2}{(x + y)^2} \times \frac{x + y}{y}$.

13. $\frac{a}{x^2 - a^2} \div \frac{a}{x(x - a)}$.

14. $\frac{x^2}{(x + y)^2} \div \frac{y}{x + y}$.

15. $\frac{a^2 + x^2}{a^2 - x^2} \times \frac{a^3 + x^3}{a^4 - x^4}$.

16. $\frac{x^3 - y^3}{x^3 + y^3} \div \frac{x - y}{x^2 - xy + y^2}$.

17. $\frac{a}{a + b} \times \frac{a}{a - b} \times \frac{a^2 - b^2}{ab}$.

18. $\frac{x^2 - x - 20}{x^2 - 25} \times \frac{x^2 - x - 2}{x^2 + 2x - 8} \div \frac{x + 1}{x^2 + 5x}$. Ans. x.

19. $\frac{ab + b^2}{a^2 - ab} \div \frac{a^3x + b^3x}{a^2c - b^2c}$. Ans. $\frac{bc(a + b)}{ax(a^2 - ab + b^2)}$.

20. $\left(\frac{1}{x^3} + 1\right) \div \left(\frac{1}{x} + 1\right)$. Ans. $\frac{x^2 - x + 1}{x^2}$.

21. $\left(\frac{x^2}{y^2} - \frac{y^2}{x^2}\right) \div \left(\frac{x}{y} + \frac{y}{x}\right)$. Ans. $\frac{x^2 - y^2}{xy}$.

22. $\left(\frac{a + b}{a} + \frac{a - b}{b}\right) \div \left(\frac{1}{a} - \frac{1}{b}\right)$. Ans. $\frac{b^2 + a^2}{b - a}$.

CHAPTER XV.

GENERALIZATION.

122. SINCE letters stand for any numbers whatever, the answer to a problem in which the given numbers are represented by letters is a general expression including all cases of the same kind. Moreover, the operations performed with letters are not, as with figures, lost in the combinations of the process, but all appear in the resulting expression. Such an expression is called a *formula*, and when expressed in words, a *rule*.

123. To illustrate the making of formulas, and their use, the following problems are solved.

1. Divide a into two parts such that the greater exceeds the less by b.

Let $x =$ the greater part;

then $x - b =$ the less part.

$\therefore$ $2x - b = a$

$$2x = a + b$$

$$x = \frac{a+b}{2}, \text{ the greater part;}$$

$$x - b = \frac{a-b}{2}, \text{ the less part.}$$

In this example a represents the sum and b the difference of two numbers, and $\frac{a+b}{2}$, $\frac{a-b}{2}$, are respectively the formulas for the two numbers. Stated in words, the formula is as follows: When the sum and difference of two numbers are given, to find the numbers we have the following

Rule.

(1) *Divide the sum plus the difference of the two numbers by two, and it will give the greater.*

(2) *Divide the sum minus the difference of the two numbers by two, and it will give the less.*

Find the numbers when

2. The sum is 14 and the difference is 6.
3. The sum is 25 and the difference is 7.
4. The sum is 50 and the difference is 8.
5. The sum is 100 and the difference is 20.
6. The sum is 18 and the difference is 12.
7. The sum is 29 and the difference is 9.
8. The sum is 105 and the difference is 15.
9. The sum is 53 and the difference is 7.
10. The sum is $2x$ and the difference is $2y$.
11. The sum is $6x^2 - 3y$ and the difference is $2x^2 + y$.
12. The sum is $3a - 6c$ and the difference is $5a + 4c$.

13. Two boys 20 rods apart walked directly toward each other till they met. If one walked 4 rods more than the other, how many rods did each walk?

14. If A can do a piece of work in a hours and B can do it in b hours, how long will it take A and B together to do the work?

Let $x =$ the required number of hours

then
$$\frac{1}{a} + \frac{1}{b} = \frac{1}{x}$$
$$bx + ax = ab$$
$$x = \frac{ab}{a+b}, \text{ the required number of hours.}$$

That is, $\frac{ab}{a+b}$ is the formula that expresses, in terms of the time in which each can do it alone, the time it will take two men together to do a given piece of work. Stated in words, the formula is as follows: When

the time it takes each of two men to do a piece of work is given, to find the time it will take the two to do it together, we have the following

Rule.

Divide the product by the sum of the numbers expressing the time it will take each to do it alone.

15. If Charles can do a piece of work in 3 days and James in 2 days, how long will it take both together to do it?

Ans. $\frac{3 \cdot 2}{3+2}$, or $1\frac{1}{5}$ days.

16. If of two pipes, one will fill a cistern in 4 hours and the other in 5 hours, how many hours will it take both running together to fill it?

17. If A can build a certain piece of wall in 7 days and B in 10 days, how many days will it take A and B together to build the wall?

SIMPLE INTEREST.

124. Though the first letters of the alphabet usually represent known numbers, and the last unknown numbers, it is often convenient in formulas to use the initial letters of the names of quantities whether known or unknown.

In the following examples in interest let p represent the principal, r the rate, t the time, i the interest, and a the amount. It must be understood that t represents the time in years, and that r is a decimal whose denominator is 100, and represents the interest of a unit of the principal for a year.

18. What is the interest of p dollars for t years at r per cent? What is the amount?

The interest = principal × time × rate,

or $$i = p\,t\,r \qquad (1)$$

The amount = principal + interest,

or $$a = p + p\,t\,r \qquad (2)$$

These formulas contain four different things, of which any one may be determined when the others are known.

Thus from (1) and (2), when a, p, and r are known, we have

$$t = \frac{i}{pr}, \text{ or } \frac{a-p}{pr}$$

Hence to find the time when the amount, principal, and rate are known we have the following

Rule.

From the amount subtract the principal, and divide the remainder by the product of the principal and the rate.

19. From equation (1) find the formula for the principal when the interest, time, and rate are given.
State the rule.

20. From equation (2) find the formula for the principal when the amount, time, and rate are given.
State the rule.

21. From equation (1) find the formula for the rate when the principal, time, and interest are given.
State the rule.

22. From equation (2) find the formula for the rate when the principal, time, and amount are given.
State the rule.

23. How long must \$200 be on interest at 6 % to gain \$36 ?

24. How long must \$500 be on interest at 4 % to amount to \$620 ?

25. At what rate must \$300 be put on interest to gain \$63 in 3 years ?

26. What principal at 5 % will gain \$400 in 8 years ?

27. How long will it take a sum of money on interest at 6 % to double itself ? At 5 % ? At 4 % ?

PRESENT WORTH AND DISCOUNT.

125. The **Present Worth** of a debt, payable at a future time without interest, is a sum of money which, put at interest, will amount to the debt at the time of its becoming due.

The *debt*, then, is an *amount*, the *present worth* is the *principal*, that is, we have the amount, time, and rate given to find the principal.

In formula (2), § 188, p stands for the present worth.

$$p + p\,t\,r = a$$

$$(1 + t\,r)\,p = a$$

$$p = \frac{a}{1 + t\,r}$$

Hence to find the present worth of an amount when the time and rate are given, we have the following

Rule.

Divide the amount by one plus the product of the time and the rate.

28. What is the present worth of $300 due in 4 years at 5 %?

29. What is the present worth of $560 due in 2 years at 6 %?

30. What is the present worth of $440 due in 2 years at 5 %?

31. What is the present worth of $1000 due in 3 months at 6 %?

The **Discount** is the interest on the present worth, and is equal to the amount due minus its present worth.

32. Find the discount on $336 due in 3 years at 4 %.

33. What is the discount on $672 due in 2 years at 6 %?

34. What is the discount on $220 due in 4 months at 6 %?

35. What is the discount on $500 due in 3 months at 4 %?

36. What is the discount on $1000 due in 2 months at 6 %?

CHAPTER XVI.

MISCELLANEOUS EXAMPLES.

Solve the equations :

1. $2x + 3 = 16 - (2x - 3)$.

2. $x - (4 - 2x) = 7(x - 1)$.

3. $5(4 - 3x) = 7(3 - 4x)$.

4. $2(x - 3) = 5(x + 1) + 2x - 1$.

5. $4(1 - x) + 3(2 + x) = 13$.

6. $5(x + 2) = 3(x + 3) + 1$.

7. $8(9 - 2x) - 17(25 - 3x) = -3$.

8. $7(8 - 3x) + 6(2x - 5) = -28$.

9. $3x - 4\{9 - (2x + 7) + 3x\} = 13$.

10. $7x - \{4x - 1 - (6x + 4)\} = 27 - 2[5x - (3x + 2)]$.

11. $(2x - 3)^2 - (2x - 7)^2 = 5(x + 3)$.

12. $(7 + x)(2 - x) = (2 - x)(5 - x) - (2x - 5)(x - 5)$.

13. $2x - [3 - \{4x + (x - 1)\} - 5] = 8$.

14. $\dfrac{x - 1}{2} + \dfrac{x - 2}{3} + \dfrac{x - 3}{4} = 10$.

15. $\dfrac{5 - 2x}{4} - \dfrac{x + 8}{3} = 1\tfrac{1}{12}$.

16. $\dfrac{x + 3}{2} - \dfrac{11 - x}{5} = \dfrac{3x - 1}{20} + 3\tfrac{1}{5}$.

17. $\tfrac{1}{4}(2x + 5) + \tfrac{1}{5}(3x - 8) = \tfrac{1}{20}(4x - 3)$.

18. $\dfrac{3x - 8}{9} - \dfrac{2x + 5}{3} + \dfrac{5x - 6}{5} = \dfrac{4x + 3}{15} - \dfrac{14}{9}$.

19. $\frac{3x}{5}+\frac{2x-4}{3}=\frac{5x}{3}-\frac{22}{3}.$

20. $\frac{x+7}{14}-\frac{x-7}{7}=\frac{x}{8}+\frac{1}{8}.$

21. $\frac{5}{6}-\frac{1}{3}\left(11-\frac{x}{2}\right)=2-\frac{x}{6}-3-\frac{x}{4}.$

22. $\frac{5}{3}-\frac{x}{3}+\frac{19}{2}=3x-\frac{2x-5}{10}-x.$

23. $\frac{5x}{6}+\frac{1}{3}\left(\frac{x}{4}-3\right)-\frac{x-12}{5}=\frac{5x}{4}-\frac{x+3}{3}.$

24. $5\frac{3}{4}-\frac{x+10}{5}=x-\frac{x}{4}-\frac{x-2}{3}.$

25. $7x+8\frac{1}{4}-\frac{3x}{2}=10\frac{1}{4}-\frac{2x}{7}+\frac{11x}{2}.$

26. $4x-\frac{x-12}{3}+5=\frac{20x+21}{4}-\frac{1}{4}.$

27. $\frac{7x+5}{7}+\frac{6x-30}{7x-7}=x+1.$

28. $\frac{284-4x}{3}-\frac{75-3x}{6}=19-\frac{22-2x}{12}.$

29. $17-\frac{3+5x}{6}-4x=\frac{3x+3}{4}-\frac{18-5x}{3}.$

30. $\frac{23-x}{4}+\frac{3x-19}{2}+10=3x-\frac{5x-38}{3}-8.$

31. $\frac{x}{21}-\frac{3(x+1)}{11}=\frac{4(x-7)}{7}-\frac{7x-17}{10}.$

32. $\frac{4}{3}-\frac{8-5x}{2}-\frac{5x}{4}=\frac{10x-7}{3}-2.$

33. $6-\frac{2}{3}(4x-1)=\frac{1}{5}(2-5x)-\frac{1}{7}(3x+2).$

34. $x-\left(\frac{x}{2}-\frac{x}{3}\right)=5.$

35. $2x-\frac{1}{2}\left(\frac{x}{3}+\frac{x}{4}\right)=7-\frac{1}{4}\left(\frac{x}{2}-\frac{x}{3}\right).$

36. $\frac{1}{2}x + \frac{1}{3}(2 - x) = \frac{1}{4}\{2x - \frac{1}{3}(5 + x)\} - \frac{1}{8}(x - 5)$.

37. $0.3x - 5 + 0.65x = 0.5x + 8.5$.

Transposing and uniting, $0.45x = 13.5$

Dividing by 0.45, $x = 30$

38. $\dfrac{0.75 + x}{0.125} = 15 - \dfrac{0.25 - x}{0.25}$.

Performing the indicated division, $6 + 8x = 15 - 1 + 4x$.

39. $1.1x - 0.25 = 0.75x + 0.8$.

40. $0.5x - 0.6x + 0.65x - 0.775x + 15 = 0$.

41. $2.5x - 1 = 0.25x + 2x + 0.2x$.

42. $1.2x + 0.05 = 0.07x + 0.3x + 16.65$.

43. $0.75x - 0.375 + 2 = x - 0.25 + 0.125x$.

44. $2.4x - \dfrac{0.36x - 0.05}{0.5} = 0.8x + 8.9$.

45. $0.15x + \dfrac{0.135x - 0.225}{0.6} = \dfrac{0.36}{0.2} - \dfrac{0.09x - 0.18}{0.9}$.

46. $a + x - b = a + b$.

47. $x - 2b = 2a - x$.

48. $ax - b = a - bx$.

49. $ax - a^3 = bx - b^3$.

50. $ax - a - b = c - bx - cx$.

51. $ax - bx - a^2 + b^2 = 0$.

52. $\dfrac{x}{a} - b = \dfrac{x}{c} + \dfrac{x}{d} + e$.

Clearing of fractions and transposing,

$$cdx - adx - acx = abcd + acde$$

$$(cd - ad - ac)x = abcd + acde$$

$$x = \frac{abcd + acde}{cd - ad - ac}$$

53. $a(x - b) + ax - b = bx - a$.

54. $a(x - 2b) + b(x - 2c) + c(x - 2a) = a^2 + b^2 + c^2$.

55. $\dfrac{x}{a} + \dfrac{x}{b} = 1$.

56. $\dfrac{ax}{b} - a^2 = b^2 - \dfrac{bx}{a}$.

57. $\dfrac{ax}{2} - \dfrac{bx}{6} + b^2 = a^2 - \dfrac{ax}{3} - \dfrac{5b + ax}{6}$.

58. $\dfrac{3}{b}(x - 2a) + \dfrac{2}{a}(x + b) = 1$.

59. $ax - \dfrac{bx + 1}{x} = \dfrac{a(x^2 - 1)}{x}$.

60. $\dfrac{x^2 - a}{bx} - \dfrac{a - x}{b} = \dfrac{2x}{b} - \dfrac{a}{x}$.

61. $\dfrac{3}{c} - \dfrac{ab - x^2}{bx} = \dfrac{4x - ac}{cx}$.

62. $am - b - \dfrac{ax}{b} + \dfrac{x}{m} = 0$.

63. $\dfrac{3ax - 2b}{3b} - \dfrac{ax - a}{2b} = \dfrac{ax}{b} - \dfrac{2}{3}$.

64. $\dfrac{ab + x}{b^2} - \dfrac{b^2 - x}{a^2 b} = \dfrac{x - b}{a^2} - \dfrac{ab - x}{b^2}$.

65. One flock of sheep consists of two sheep more than half of another flock. They both together amount to 101 sheep. How many are there in each?

66. John has 80 cents and James has 15. How many cents must John give to James in order that he may have just four times as many as James?

67. A can do a piece of work in 15 hours, but with the help of B he gets it done in 5 hours. In what time can B do it alone? 7 ½

68. A spends $\frac{1}{3}$ of his income in board, $\frac{1}{8}$ in clothes, $\frac{1}{10}$ in sundries, and has \$212 left. What is his income? $480.

69. $\frac{2}{5}$ of A's money is equal to B's, and $\frac{7}{9}$ of B's is equal to C's. In all they have \$770. How much has each?

70. What is the property of a person whose income is \$430, when he has $\frac{2}{3}$ of it invested at 4 %, $\frac{1}{4}$ at 3 %, and the remainder at 2 %? Ans. \$12000.

71. A testator left $\frac{1}{2}$ of his estate to his widow, $\frac{1}{6}$ to each of his two sons, $\frac{1}{12}$ to his servant, and the residue, \$600, to charities. What was his whole estate?

72. A, B, C, and D divide \$2520 as follows: C has \$360, B as much as C and D together, and A \$1000 less than twice as much as B. What is the share of each?

73. In a mixture of oats and barley, the oats are 25 bushels more than half of the mixture, and the barley 5 bushels less than a third of it. How many bushels are there of each?

74. A starts from a certain place and travels at the rate of 7 miles in 3 hours; B starts from the same place 6 hours after A, and travels in the same direction at the rate of 5 miles in 2 hours. How far will A travel before he is overtaken by B?

75. A market-woman bought a certain number of eggs at the rate of 5 for 2 cents, and sold half of them at 2 for a cent, and half of them at 3 for a cent, and gained four cents. How many eggs did she buy? Ans. 240.

76. An officer can form the men of his regiment into a hollow square 6 deep. The number of men in the regiment is 1320. Find the number of men in the front of the hollow square.

Ans. 61.

77. An officer can form his men into a hollow square 6 deep, and also into a solid square 36 in front. Find the number of men in the front of the hollow square. Ans. 60.

78. A cistern is filled by each one of its three pipes, running alone, in a, b, and c minutes respectively. How long will it take the pipes to fill it when they are all running together?

79. $\dfrac{2x}{x-6} - 1 = \dfrac{13}{x-6}.$

80. $\dfrac{x}{x-1} + \dfrac{x+1}{x} = 2.$

81. $\dfrac{3}{x} - \dfrac{2}{x+1} = \dfrac{5}{4(x+1)}.$

82. $\dfrac{3}{2x-5}-\dfrac{1}{x}=\dfrac{12}{x(2x-5)}$.

83. $\dfrac{6}{x-1}-\dfrac{4x}{x^2-1}=\dfrac{3}{x+1}$.

84. $\dfrac{7}{x-1}=\dfrac{6x+1}{x+1}+\dfrac{3+6x^2}{1-x^2}$.

85. $\dfrac{4}{x}+\dfrac{7}{x+1}=\dfrac{37}{x^2+x}$.

86. $\dfrac{4}{x+2}+\dfrac{7}{x+3}=\dfrac{37}{x^2+5x+6}$.

87. $\dfrac{4}{x^2-1}+\dfrac{1}{1+x}=\dfrac{1}{1-x}$.

88. $\dfrac{1}{x-5}+\dfrac{3}{2x-6}=\dfrac{5}{x^2-8x+15}$.

89. $\dfrac{x+a}{x-b}+\dfrac{x+b}{x-a}=2$.

90. $\dfrac{1}{x-a}-\dfrac{1}{x-b}=\dfrac{a-b}{x^2-ab}$.

91. $\dfrac{x+a}{x-a}-\dfrac{x-b}{x+b}=\dfrac{2(a+b)}{x}$.

92. $\begin{cases}5x-3y=-5.\\2x-5y=-40.\end{cases}$

93. $\begin{cases}8x-21y=5.\\6x+14y=-26.\end{cases}$

94. $\begin{cases}\dfrac{x}{2}+\dfrac{y}{3}=1.\\[2mm]\dfrac{x}{4}-\dfrac{2y}{3}=3.\end{cases}$

95. $\begin{cases}\dfrac{x}{5}+5y=-4.\\[2mm]\dfrac{y}{5}+5x=4.\end{cases}$

96. $\begin{cases}x+\dfrac{3}{y}=\dfrac{7}{2}.\\[2mm]3x-\dfrac{2}{y}=\dfrac{26}{3}.\end{cases}$

97. $\begin{cases}\dfrac{1}{y}+\dfrac{1}{x}=8.\\[2mm]\dfrac{1}{y}-\dfrac{1}{x}=4.\end{cases}$

98. $\begin{cases}\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{6}.\\[2mm]\dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{6}.\end{cases}$

99. $\begin{cases} \dfrac{2}{x} + \dfrac{7}{y} = 17. \\ \dfrac{7}{x} + \dfrac{2}{y} = 37. \end{cases}$

100. $\begin{cases} x + y = a. \\ x - y = b. \end{cases}$

101. $\begin{cases} a\,x + b\,y = c. \\ m\,x + n\,y = d. \end{cases}$

102. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = 1. \\ \dfrac{x}{b} + \dfrac{y}{a} = 1. \end{cases}$

103. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = m. \\ \dfrac{b}{x} + \dfrac{a}{y} = n. \end{cases}$

104. $\begin{cases} 3\,x - 2\,y + 3\,z = 8. \\ 3\,x + 2\,y + 2\,z = 13. \\ 5\,x - 2\,y + 2\,z = 7. \end{cases}$

105. $\begin{cases} 2\,x - 3\,y + 4\,z = 9. \\ 3\,x - 5\,y - 2\,z = -4. \\ 2\,x - 6\,y - 5\,z = -18. \end{cases}$

106. $\begin{cases} x + y = 2\tfrac{1}{2}. \\ y + z = -2. \\ x + z = -\tfrac{1}{2}. \end{cases}$

107. $\begin{cases} x + y = a. \\ y + z = b. \\ x + z = c. \end{cases}$

108. $\begin{cases} a\,x + b\,y = 1. \\ b\,y + c\,z = 1. \\ a\,x + c\,z = 1. \end{cases}$

109. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 2. \\ \dfrac{1}{y} + \dfrac{1}{z} = 3. \\ \dfrac{1}{x} + \dfrac{1}{z} = 4. \end{cases}$

110. Two brothers received by will equal sums of money. After several years the elder had increased his by 25 %, while the younger had lost 50 % of his. Then the elder had \$2400 more than the younger. What sum did the younger have?

111. A workman agrees to work at \$2.50 a day, and to forfeit \$3 for every day he does not work. At the end of 33 days he receives \$44. How many days did he work?

112. How many days must the workman in Ex. 111 be away from his work to receive nothing?

113. A railroad train ran a certain distance. If the rate had been $\frac{4}{5}$ of its actual rate, the time would have been $27\frac{1}{2}$ hours; and if the rate had been 24 kilometers more an hour, the train would have completed $\frac{4}{7}$ of the distance in 8 hours. Find the distance. Ans. 924 kilometers.

114. A man paid \$1725 with gold eagles and silver dollars. There were 54 eagles as often as there were 35 silver pieces. How many coins of each kind were there?

115. A farmer paid \$1600 for 24 acres of land. One part gave him a revenue of $4\frac{1}{4}$ %, and the other of $3\frac{1}{2}$ %, and the total revenue was \$63. Find the areas of the parts.

116. A man wills his property to his children, in such a way that the eldest is to have \$500 and an eighth of the remainder, the second \$1000 and an eighth of the second remainder, the third \$1500 and an eighth of the third remainder, and so on to the youngest. The legacies are thus equal to one another. Find the value of the property and the number of children.

Ans. \$24500; 7 children.

117. A merchant sold a piece of cloth so as to make 12 % of the sale. The profit is \$200 more than $\frac{1}{10}$ of the price paid. What was the price paid?

118. The following is said to be inscribed on the tombstone of Diophantus of Alexandria: "One sixth of his life he spent in boyhood; one twelfth in youth; he was then married, and passed 5 years more than one seventh of his life with his wife before having a son, whom he survived 4 years, and who, dying, was one half the age of the father." How old was Diophantus at his death?

119. There is a certain number of two figures whose sum is 10. If the order of the figures is reversed, the number thus formed is 34 more than three times the first number. What is the number?

120. A vase full of water weighs 12065 grams; full of oil, 11785 grams. The oil weighs 0.912 as much as the water. What is the weight of the vase? Ans. $8883\frac{2}{11}$ grams.

121. $\frac{1}{3}$ of the value of a piece of silk is equal to $\frac{1}{5}$ of the value of a piece of woollen. The difference of the two values is \$38.40. The length of the woollen is $\frac{10}{3}$ of the length of the silk, and a yard of the woollen is worth \$1.60. What is the length of each? Ans. Silk, 18 yards; woollen, 60 yards.

122. A can full of milk weighs 2520 grams, and full of oil, 2440. The milk weighs 0.95 as much as pure water, and the oil 0.9. Find the weight of the can. Ans. 1000 grams.

123. I am twice as old as you were when I was as old as you now are. When you are as old as I am now, the sum of our ages will be 117 years. What is my age ?

124. A man placed at interest \$5068.80 at 5 %, and 7 months later \$4928 at 6 %. When will the interest on the two sums be equal ?

125. There are three casks. The second contains $\frac{7}{9}$ as much as the first, and the third $\frac{3}{4}$ as much as the second, and 50 liters less than the first. Find the capacity of each cask.

126. A poulterer sold all his eggs to 4 persons : to the first $\frac{4}{5}$ of all he had plus $\frac{4}{5}$ of an egg ; to the second $\frac{4}{5}$ of the remainder plus $\frac{4}{5}$ of an egg ; and so on to the rest, to each $\frac{4}{5}$ of the remainder plus $\frac{4}{5}$ of an egg. No egg was broken. Find the number of eggs sold, and the number to each.

Ans. 624 eggs. To 1st, 500; 2d, 100 ; 3d, 20 ; 4th, 4.

127. From Paris to Lyons is 512 kilometers. A train starts from Paris for Lyons at 8 h. 45 m. A. M., at the rate of 30 kilometers an hour ; another train starts from Lyons for Paris at 1 h. 15 m. P. M., at the rate of 28 kilometers an hour. How far from Lyons and at what time will the trains pass each other ?

Ans. 182 kilometers; 7 h. 45 m. P. M.

128. A farmer employs a man and a boy. To the man he pays twice as much daily wages as to the boy. For 15 days' work he gives the man \$37.50 and 10 gallons of maple syrup, and to the boy for 12 days' work, \$16.50 and 2 gallons of maple syrup. What is the price of the syrup a gallon ? Ans. \$0.75.

129. For 15 pounds of coffee and 12 pounds of sugar one pays \$6.21 ; and for 17 pounds of coffee and 14 pounds of sugar, \$7.07. Find the price of each a pound.

130. A certain capital is put at interest for a year. If the rate were 1 % more and the capital \$200 more, the interest would be \$16 more; if the rate were 2 % more and the capital \$300 more, the interest would be \$30 more. Find the rate and the capital. Ans. Rate, 4 %; capital, \$600.

131. Find two numbers such that a third of the first exceeds by $\frac{1}{4}$ a fourth of the second, and four thirds of the first minus two fifths of the second is equal to three fourths of the first plus nineteen fortieths of the second.

132. Two barrels full of oil that is worth 17 cents a quart are sold for prices that differ by \$4.08; and $\frac{5}{6}$ of the capacity of one barrel is equal to $\frac{10}{13}$ of the capacity of the other. Find the capacity of each barrel. Ans. 288 quarts and 312 quarts.

133. Two masses of iron are such that $\frac{2}{5}$ of the first weighs 96 pounds less than $\frac{3}{4}$ of the second, and $\frac{5}{8}$ of the second as much as $\frac{4}{9}$ of the first. Find the weight of each.

134. Two boys work together, and the daily wages of the first are $\frac{4}{3}$ the daily wages of the second. The first, who works 5 days more than the second, receives \$20, and the second \$12. What are the daily wages and the number of days for each?

135. Two cloaks are made for two sisters. For the elder it takes 3 yards of woollen and 8 yards of silk lining, and for the younger 2 yards of woollen and 5 yards of silk lining. The cloak for the elder cost \$9.75, and the cloak for the younger \$6.25. What was the price for a yard of woollen and for a yard of silk?

136. If the pages of this book had an average of 3 lines more on a page and 4 letters more in a line, they would have 224 more letters on a page; but if they had 2 lines less on a page and 3 letters less in a line, they would have 145 letters less on a page. How many lines are there on a page, and how many letters in a line?

ANSWERS.

Answers which, if seen, would nullify the utility of the example, are omitted.

Article 12, pages 7, 8.

6. 110.

7. 8.

8. 40.

9. 43. 301.

10. Horse, \$210. Carriage, \$110.

12. A, $\$2015\frac{1}{3}$. B, $\$4530\frac{2}{3}$.

14. 1st, 16. 2d, 8. 3d, 24.

15. A, \$69. B, \$46. C, \$23.

16. D, $5\frac{5}{9}$ yrs. S, $11\frac{1}{9}$ yrs. F, $33\frac{1}{3}$ yrs.

17. A, \$220. B, \$110. C, \$170.

18. 75.

Articles 13–16, pages 9–11.

4. 2.

5. 1.

6. 3.

7. 5.

10. 16.

11. 35.

12. 12.

14. 10.

15. 1.

16. 18.

17. 20.

18. 30.

19. 8.

20. 24.

21. 18.

22. Transpose $7\frac{1}{2}$ and unite it with $9\frac{1}{2}$; also transpose the $\frac{x}{2}$ and unite it with $-\frac{11x}{2}$; transpose $-\frac{2x}{7}$. We have $2=\frac{2x}{7}$, and $x=7$.

24. 120.

Article 18, page 12.

6. 23.

7. 63.

Article 19, pages 13, 14.

2. 6.
3. 12.
5. 9.
6. 17.
7. 14.
8. 2.
9. 1.
10. 7.
12. 30.
13. 7.
14. 65.
15. 7.
17. 1.
18. 5.
19. 11.
20. 3.
21. 19.
22. 4.

Article 21, pages 16–18.

1. $2x - 10.$
2. $x + 12.$
3. $\begin{cases} \text{John, } (x+4) \text{ years.} \\ \text{James, } (x-5) \text{ years.} \end{cases}$
4. $\begin{cases} \text{A, } \$(2x+40). \\ \text{B, } \$(4x+10). \\ \text{Both, } \$(6x+50). \end{cases}$
5. $\begin{cases} \text{Robert, } 2x+4. \\ \text{George, } 55-4x. \end{cases}$
6. $\begin{cases} \text{A, } \$(8x-10). \\ \text{B, } \$(4x+10). \end{cases}$
7. $\$(8x - 3500).$
8. $\$(16y - 62).$
9. $2n + 2.\ \ 2n - 2.$
10. $2n+1.\ \ 2n+3.\ \ 2n-3.$
11. $6n - 3.$
12. $6x$ years.
13. $(m + x + n)$ years.
14. $(2n + 8)$ years.
15. $(tx + y)$ years.
16. $2a.$
17. $om + cm.$
18. $\begin{cases} \dfrac{h}{m} \text{ hours a mile.} \\ \dfrac{m}{h} \text{ miles an hour.} \end{cases}$
19. $3x.$
20. $\dfrac{5m}{6h}$ miles.
21. $\dfrac{ab}{25}$ hours.
22. $\dfrac{y}{xz}$ days.
23. $\begin{cases} (a+b) \text{ miles.} \\ 5(a+b) \text{ miles.} \\ \dfrac{a+b}{4} \text{ miles.} \end{cases}$
24. $\begin{cases} (a-b) \text{ miles.} \\ \dfrac{7(a-b)}{3} \text{ miles.} \end{cases}$
25. $\dfrac{1}{d}.\ \ \dfrac{2}{d}.\ \ \dfrac{b}{d}.$
26. $\dfrac{y}{x}.$

27. $\begin{cases} \dfrac{1}{a} + \dfrac{1}{b} = \dfrac{a+b}{a\,b}. \\ \dfrac{a\,b}{a+b} \text{ days.} \end{cases}$

29. $\begin{cases} \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = \dfrac{ab+bc+ac}{a\,b\,c}. \\ \dfrac{a\,b\,c}{a\,b + b\,c + a\,c} \text{ days.} \end{cases}$

28. $\dfrac{x}{y}$. $\dfrac{y}{x}$ hours.

Article 22, pages 19–28.

4. 12 years.
5. Horse, $270. Carriage, $90.
7. 68.
8. 7.
10. Sheep, $6. Lamb, $3.
11. 1st, $18. 2d, $27. 3d, $36.
14. 5.
15. A's, $28. B's, $22.
17. Eldest, $8000. 2d, $6000. 3d, $5000.
18. Men, 18. Women, 22. Children, 50.
19. A, $46. B, $16.
21. 60 miles.
31. A, $175. B, $225. C, $360.
35. A, $18. B, $24.
36. 30 gallons.
41. 3 years.
44. 1st, 30. 2d, 25.
60. $8\frac{1}{2}$ years.
62. 48.
63. 11 years.
64. $7.
65. 90.
67. $300.
68. $4000.
69. $9\frac{1}{2}$.
70. $7\frac{1}{4}$.
71. 75 miles.
72. 40 men, 15 boys.
73. Father, 36 years. Son, 6 years.
74. 1st, $28560. 2d, $4896.

Article 24, pages 30, 31.

2. $\begin{cases} x = 2. \\ y = 3. \end{cases}$
3. $\begin{cases} x = 3. \\ y = 7. \end{cases}$
4. $\begin{cases} x = 1. \\ y = 6. \end{cases}$
5. $\begin{cases} x = 2. \\ y = 5. \end{cases}$
6. $\begin{cases} x = 4. \\ y = 9. \end{cases}$
7. $\begin{cases} x = 11. \\ y = 13. \end{cases}$

8. $\begin{cases} x = 4. \\ y = 3. \end{cases}$

9. $\begin{cases} x = 5. \\ y = 7. \end{cases}$

10. $\begin{cases} x = 1. \\ y = 1. \end{cases}$

11. $\begin{cases} x = 8. \\ y = 9. \end{cases}$

12. $\begin{cases} x = 3. \\ y = 11. \end{cases}$

13. $\begin{cases} x = 5. \\ y = 13. \end{cases}$

15. $\begin{cases} x = 5. \\ y = 4. \end{cases}$

16. $\begin{cases} x = 2. \\ y = 7. \end{cases}$

17. $\begin{cases} x = 1. \\ y = 2. \end{cases}$

18. $\begin{cases} x = 4. \\ y = 3. \end{cases}$

19. $\begin{cases} x = 15. \\ y = 12. \end{cases}$

20. $\begin{cases} x = 5. \\ y = 7. \end{cases}$

Article 25, pages 31, 32.

1. $\begin{cases} x = 2. \\ y = 7. \end{cases}$

2. $\begin{cases} x = 9. \\ y = 12. \end{cases}$

3. $\begin{cases} x = 6. \\ y = 21. \end{cases}$

4. $\begin{cases} x = 12. \\ y = 16. \end{cases}$

5. $\begin{cases} x = 15. \\ y = 16. \end{cases}$

6. $\begin{cases} x = 14. \\ y = 10. \end{cases}$

7. $\begin{cases} x = 10. \\ y = 20. \end{cases}$

8. $\begin{cases} x = 15. \\ y = 30. \end{cases}$

9. $\begin{cases} x = 11. \\ y = 3. \end{cases}$

10. $\begin{cases} x = 8. \\ y = \frac{1}{2}. \end{cases}$

11. $\begin{cases} x = 11. \\ y = 5. \end{cases}$

12. $\begin{cases} x = 7. \\ y = 10. \end{cases}$

13. $\begin{cases} x = 44. \\ y = 20. \end{cases}$

14. $\begin{cases} x = 21. \\ y = 20. \end{cases}$

Article 26, page 35.

8. $\begin{cases} x = 2. \\ y = 3. \\ z = 4. \end{cases}$

9. $\begin{cases} x = 1. \\ y = 2. \\ z = 3. \\ w = 4. \end{cases}$

10. $\begin{cases} x = 6. \\ y = 9. \\ z = 12. \end{cases}$

11. $\begin{cases} x = 6. \\ y = 2. \\ z = 9. \end{cases}$

12. $\begin{cases} x = \frac{1}{3}. \\ y = \frac{1}{4}. \\ z = \frac{1}{5}. \end{cases}$

14. $\begin{cases} x = 1. \\ y = 1. \\ z = 1. \end{cases}$

15. $\begin{cases} x = 2. \\ y = 3. \\ z = 6. \end{cases}$

16. $\begin{cases} x = \frac{1}{4}. \\ y = \frac{1}{3}. \\ z = \frac{1}{2}. \end{cases}$

Article 27, pages 37–40.

8. 36.

9. Man's, $2 a day.
   Son's, $1 a day.

10. Men, 14.
    Boys, 5.

13. $\frac{2}{7}$.

14. $\frac{1}{4}$. $\frac{1}{5}$. $\frac{1}{6}$.

16. 15. 70. 72.

20. 5 cent pieces, 6.
    10 cent pieces, 5.

21. Peter, $1.
    John, $2.

22. $\frac{3}{7}$.

23. $\frac{5}{9}$.

24. $\frac{2}{9}$.

25. A, 2 days.
    B, 3 days.
    C, 4 days.

26. A, 10 days.
    B, 6 days.

27. A, 60 days.
    B, 30 days.

28. A, 4 days.
    B, 5 days.

29. 216.

30. 83.

31. 570.

32. 1st, $200.
    2d, $300.
    3d, $400.

Article 34, page 46.

6. 12 miles east.

7. 10 miles west.

8. — $6, or $6 in debt.

Article 41, page 47.

3. $20xy$.

5. $14a$.

7. $-16ax$.

9. $16bx$.

10. $30ax$.

12. $-17bc$.

13. $-10bcd$.

Article 42, pages 48, 49.

3. $13x$.

5. $2xyz$.

8. $3x$.

10. $22az$.

11. $-4bc$.

12. $9bc$.

Article 46, page 51.

2. $14a - b - 7c + d$.

3. 1.

4. $9b - 4c + 2x + 3y + z$.

5. $9x - y$.

6. $2a + 4b$.

Article 47, page 52.

8. $-2a - 16b - 23$.

11. 0.

15. 0.

Article 51, page 55.

2. 64 years. 4. $84°$. 5. $19°$.

Article 52, page 56.

5. $8cd$.
8. $9a$.
9. $-14ab$.
10. $6a-8b-5c$.
11. $3x+3y+14z$.
12. $x-4y-15z$.
13. $2cd-2ac-9bd$.
14. $8x-8y+3$.
15. $2a+7b-3c$.
16. $7a-8b-c+2d$.
17. $-3a+4b-c$.
18. $4a+4b-19$.
19. $20a-26$.

Article 54, page 58.

1. $4a-b$.
2. $4b$.
3. $2a$.
4. $2x$.
5. $-x$.
6. b.
7. $8a-3b$.
8. $4a-4b-c$.

Article 55, page 58.

1. $a+b-(c-d)$.
2. $a-(b+c+d)$.
3. $-(a+b-c)-(d-e+f)$.
4. $-a-(b+c-d-e)$.

Article 67, pages 62, 63.

8. a^7.
9. x^7.
10. $-c^5$.
11. $-x^8$.
12. a^{13}.
13. $8a^{m+n}$.
14. a^{m+n}.
15. $40x^5y^7$.
16. $-35a^6b^8$.
17. $12x^7y^3$.
18. $48a^5x^3y^3$.
19. $36x^5y^3z^3$.
20. $-200a^7b^9$.
21. $300a^5b^4c^6$.
22. $60a^2b^3c$.
23. $-90a^6b^4c^3d^3$.
24. $42x^9y^6z^6$.
25. $12(x+y)^3$.

Article 68, pages 63, 64.

3. $2a^4 + 6a^3 + 8a^2$.
4. $a^3 b - 2a^2 b^2 + a b^3$.
5. $15x^5 - 25x^4 + 10x^3$.
6. $a^3 b c - a b^3 c - a b c^3$.
7. $-10x^7 + 15x^6 + 5x^4$.
8. $15x^3 y^2 - 18x^2 y^3 + 24x^3 y^3$.
9. $3a^5 b - 9a^4 b^3 + 3a^2 b^4$.
10. $-16x^6 y^{12} + 14x^4 y^8 - 8x^3 y^6 + 6x^2 y^4$.
11. $x^5 y - 3x^4 y^2 + 3x^3 y^3 - x^2 y^4$.
12. $-5x^2 y^3 z^2 + 3x^2 y^2 z^3 - 8x^3 y^2 z^2$.

Article 69, page 65.

4. $x^2 + (a + b)x + a b$.
5. $4a^2 + 8ab + 3b^2$.
6. $6a^3 + 7a^2 - 10a + 25$.
7. $a^4 + a^2 b^2 + b^4$.
9. $a^4 + 4a^2 x^2 + 16x^4$.
11. $64a^3 - 27b^3$.
13. $4x^5 - x^3 + 4x$.
15. $a^6 + a^3 b^3$.
17. $-x^4 + 4x^3 y - x^2 y^2 - 4x y^3 - y^4$.
19. $x^4 - 2x^2 y^2 + y^4$.

Article 71, page 67.

7. $a c$.
8. $-a$.
9. $a x^3$.
10. $-7abc$.
11. $-8b^2 x$.
12. $4ac$.
13. a.
15. a^{m-2}.
16. a^{m-n}.
17. a^{m-1}.
18. a^p.
19. a^q.
20. $2a^{m-1} b^{n-2}$.
22. $3(a + b)^3$.
23. $-3(x + y)^2$.
24. $5(x + y)$.
25. $-3(x - z)^3$.
26. $3(a + 2)^5$.
27. $-3(1 - x)^4$.

Article 72, page 68.

4. $a + b$.
5. $2a^2 - ab$.
6. $x^4 - 7x^3 + 4x^2$.
7. $-3x^2 + 5x$.
8. $3x^3 + 4x$.
9. $a^2 + ab + b^2$.
10. $a^2 - ab + b^2$.
11. $x^3 - x^2 + x - 1$.
12. $a + b + c$.
13. $a^2 z^2 + a z + 1$.
14. $-2x^2 + 3x - a$.
15. $3x^3 y - 4x^2 y^2 + 6x y^3$.
16. $-3x^2 + 4xy - 6y^2$.
17. $-z^2 + b x z^3 - b^2 x^2 z^4$.

Article 73, page 71, 72.

9. $3x + 1$.
12. $x^2 - 3x + 9$.
14. $x^2 + xy + y^2$.
15. $x^2 - xy + y^2$.
16. $x^2 - y^2$.
17. $x^2 - x + 1$.
19. $x^2 + y^2$.
21. $3x^2 - 4x - 3$.
23. $2y^2 - 6y - 4$.
25. $x^2 - 3xy - y^2$.
27. $2a^2 - 4ab - 5b^2$.
29. $x^2 - 2xy + 4y^2$.
31. $a^2 + b^2 + c^2 + d^2$.
32. $x^2 + 2x + 3$.
33. $a^2 - 3ab + b^2$.
34. $a^3 - 3a^2b + 3ab^2 - b^3$.

Article 74, page 73.

1. $x^2 + 2xy + y^2$.
2. $x^2 + 2x + 1$.
3. $x^2 + 4x + 4$.
4. $9x^2 + 6x + 1$.
5. $a^2x^2 + 2ax + 1$.
6. $4x^2 + 4ax + a^2$.
7. $a^4 + 2a^2b^2 + b^4$.
8. $9x^2 + 30xy + 25y^2$.
9. $4x^2 + 12xy + 9y^2$.
10. $x^6 + 2x^3y^3 + y^6$.
11. $9x^4 + 6x^2y + y^2$.
12. $4a^2 + 20ab + 25b^2$.
13. $9x^2 + 24xy + 16y^2$.
14. $25x^2 + 10xy + y^2$.

Article 75, page 74.

1. $x^2 - 2ax + a^2$.
2. $x^2 - 2x + 1$.
3. $x^2 - 4x + 4$.
4. $4x^2 - 4ax + a^2$.
5. $a^4 - 2a^2b^2 + b^4$.
6. $4x^2 - 12xy + 9y^2$.
7. $16x^2 - 24xy + 9y^2$.
8. $x^6 - 2x^3y^3 + y^6$.
9. $a^2b^2 - 2abxy + x^2y^2$.
10. $4x^4 - 4x^2y + y^2$.
11. $25z^2 - 30zy + 9y^2$.
12. $9a^2 - 30ax + 25x^2$.
13. $25x^2 - 20xz + 4z^2$.
14. $49z^2 - 14zy + y^2$.
15. $9a^2b^2 - 12abc + 4c^2$.
16. $4a^2c^2 - 4abc + b^2$.
17. $16x^2 - 32xy + 16y^2$.
18. $9z^2 - 12zy + 4y^2$.
19. $25a^2 - 50ab + 25b^2$.
20. $16x^2 - 48xy + 36y^2$.
21. $9a^2 - 42ab + 49b^2$.
22. $4z^2 - 32zy + 64y^2$.

23. $x^2 - 18\,x\,y + 81\,y^2$.
24. $4\,c^2 - 20\,c\,d + 25\,d^2$.
25. $25\,a^2 - 40\,a\,b + 16\,b^2$.
26. $9\,a^4 - 6\,a^2\,b^2 + b^4$.
27. $16\,a^4\,b^4 - 8\,a^2\,b^3\,c + b^2\,c^2$.
28. $36\,x^4 - 60\,x\,y + 25\,y^4$.
29. $49\,a^6 - 42\,a^3\,b^3 + 9\,b^6$.
30. $81\,x^4 - 18\,x^2\,z^2 + z^4$.

Article 76, page 75.

1. $x^2 - y^2$.
2. $x^2 - 9$.
3. $a^2 - 1$.
4. $x^4 - y^4$.
5. $4\,a^2 - 9\,b^2$.
6. $4\,a^2 - b^2$.
7. $9 - x^2$.
8. $25 - y^2$.
9. $4\,y^2 - 49$.
10. $36\,a^2 - 36$.

Article 77, page 75.

2. 9801.
3. 9604.
4. 2304.
5. 2209.
6. 249001.
7. 243049.
8. 998001.
9. 982081.

Article 78, page 76.

5. $x^2 + 9\,x + 14$.
6. $x^2 - 9\,x + 14$.
7. $x^2 + 5\,x - 14$.
8. $x^2 - 5\,x - 14$.
9. $a^2 - a - 20$.
10. $y^2 - 8\,x\,y + 15\,x^2$.
11. $x^2 - 2\,a\,x - 8\,a^2$.
12. $x^2 + 4\,x - 5$.
13. $x^2 + 7\,x - 8$.
14. $x^2 + 6\,a\,x + 8\,a^2$.
15. $a^2 + 2\,a\,b - 15\,b^2$.
16. $x^2 + a\,x - 30\,a^2$.

Article 79, page 77.

1. $x^2 + 4\,x\,y + 4\,y^2$.
2. $x^2 - 4\,x\,y + 4\,y^2$.
3. $a^2 + 6\,a\,c + 9\,c^2$.
4. $a^2 - 6\,a\,c + 9\,c^2$.
5. $4\,a^2 + 4\,a\,x + x^2$.
6. $4\,a^2 - 4\,a\,x + x^2$.
7. $9\,a^4 + 6\,a^2\,y + y^2$.
8. $9\,a^4 - 6\,a^2\,y + y^2$.
9. $4\,a^2\,x^2 + 12\,a\,x\,y + 9\,y^2$.
10. $4\,a^2\,x^2 - 12\,a\,x\,y\,z + 9\,y^2\,z^2$.
11. $4\,x^4 + 12\,x^2\,y^2 + 9\,y^4$.
12. $4\,x^4 - 12\,x^2\,y^2 + 9\,y^4$.
13. $x^2 - 100$.
14. $x^4 - 144$.

15. $x^2 - 100\,a^2$.
16. $a^4 - 81\,x^4$.
17. $4\,a^2 - 9\,b^2\,c^2$.
18. $a^2 + 2\,a\,b + b^2 - c^2$.
19. $a^2 - b^2 - 2\,b\,c - c^2$.
20. $a^2 - b^2 + 2\,b\,c - c^2$.
21. $a^2 - 2\,a\,b + b^2 - c^2$.
22. $a^2 + 2\,a\,c + c^2 - b^2$.
23. $a^4 + a^2 + 1$.
24. $a^2 + 4\,a\,b + 4\,b^2 - 9\,c^2$.
25. $a^2 + 6\,a\,c + 9\,c^2 - 4\,b^2$.
26. $a^2+b^2+c^2+2ab+2ac+2bc - d^2$.
27. $x^8 - y^8$.
28. $x^8 + y^8$.
29. $x^4 - a^4$.
30. $x^4 - 81$.
31. $16\,x^4 - 1$.
32. $1 - a^4\,x^4$.
33. $x^8 - a^8$.
34. $x^{16} - 1$.

Article 86, page 79.

3. $a,\ x + a$.
4. $x,\ x^2 - 3\,a$.
5. $x^2,\ x - 1$.
6. $2,\ a,\ 1 - a$.
7. $3,\ a,\ a + 2\,b$.
8. $5,\ 3 + 5\,x^2$.
9. $7, 1 - 5\,x$.
10. $13,\ x - 3\,y$.
11. $5,\ 3\,z + 7\,y$.
12. $a,\ b,\ 1 - x$.
13. $2,\ x,\ 2\,x - 1$.
14. $2,\ a^3,\ 3 + a + 2\,a^2$.
15. $3,\ x^2,\ x^2 - x\,y + 2\,y^2$.
16. $7,\ a,\ 1 - a^2 + 2\,a^3$.
17. $5,\ x^3,\ x^2 - 2\,a^2 - 3\,a^3$.
18. $19,\ a^3,\ x^2,\ 2\,x^3 + 3\,a$.
19. $x^3,\ x^6 - 1$.
20. $x^n,\ 1 - x^2$.
21. $2,\ a^n,\ 1 + 2\,a - 3\,a^2$.
22. $3,\ x^3,\ y,\ 1 - 2\,x\,y + 4\,x^2y^4$.
23. $7,\ a^2,\ b,\ a^2\,b^2 + 2\,a\,b - 4$.
24. $2,\ 2,\ x,\ y,\ 1 - 6xy - 9x^2y^2$.
25. $5,\ b,\ b^2 - 2\,b + 3$.
26. $3,\ 3,\ x,\ y,\ 1 - 3xy + 5x^2y^2$.

Article 87, page 80.

3. $a + b,\ a + b$.
4. $a + 3\,b,\ a + 3\,b$.
5. $a - 3\,b,\ a - 3\,b$.
6. $x - 5\,y,\ x - 5\,y$.
7. $2\,x + 3\,y,\ 2\,x + 3\,y$.
8. $9\,x - 2\,y,\ 9\,x - 2\,y$.
9. $5\,a\,b - c,\ 5\,a\,b - c$.
10. $x - a\,b\,c,\ x - a\,b\,c$.
11. $a\,x + 2\,b\,y,\ a\,x + 2\,b\,y$.
12. $1 - 2\,x,\ 1 - 2\,x$.

Article 88, page 81.

2. $x + a,\ x - a.$
3. $x + 1,\ x - 1.$
4. $2x + 1,\ 2x - 1.$
5. $a + 2b,\ a - 2b.$
6. $3x + y,\ 3x - y.$
7. $1 + 5x,\ 1 - 5x.$
8. $7 + c,\ 7 - c.$
9. $3 + a,\ 3 - a.$
10. $x + 9,\ x - 9.$
11. $3x + 4y,\ 3x - 4y.$
12. $1 + ab,\ 1 - ab.$
13. $5 + 8x,\ 5 - 8x.$
14. $x^2 + 4b,\ x^2 - 4b.$
15. $ab + 3x^2,\ ab - 3x^2.$
16. $9x^3 + 5a,\ 9x^3 - 5a.$
17. $11a^2 + b,\ 11a^2 - b.$
18. $a^5 + x^2,\ a^5 - x^2.$
19. $a^3 + x,\ a^3 - x.$

Article 89, page 81.

2. $a + b + c,\ a + b - c.$
3. $a - b + c,\ a - b - c.$
4. $x + y + z,\ x - y - z.$
5. $x + y - z,\ x - y + z.$
6. $x + y + 2x,\ x + y - 2x.$
7. $x + 2y + a,\ x + 2y - a.$
8. $a + b + 2c,\ a + b - 2c.$
9. $1 + a - b,\ 1 - a + b.$
10. $a - 2x + b,\ a - 2x - b.$
11. $2x - 3a + 3c,\ 2x - 3a - 3c.$
12. $a + b + c + d,\ a + b - c - d.$
13. $a - b + c - d,\ a - b - c + d.$
14. $4a + x + b + y,\ 4a + x - b - y.$
15. $5x + y + 1,\ 5x + y - 1.$
17. $2x + y,\ y.$
18. $y,\ 2x - y.$
19. $x + 5y,\ x + y.$
20. $-2,\ 2,\ a,\ b.$
21. $2,\ 2,\ a,\ b.$
22. $5,\ a,\ 2 - a.$
23. $12x - 1,\ 2x + 7.$
24. $2,\ 2,\ a,\ b - 3.$

Article 90, page 82.

4. $a - b + x,\ a - b - x.$
5. $2a + b + 3c,\ 2a + b - 3c.$
6. $c + d + 1,\ c + d - 1.$

7. $x+a+b,\ x-a-b.$
8. $x+y+2xy,\ x+y-2xy.$
9. $x^2+x+1,\ x^2-x-1.$
10. $1+a+b,\ 1-a-b.$
11. $a^2+1-x,\ a^2-1+x.$
12. $a+b+c-d,\ a+b-c+d.$
13. $2,\ x+1,\ 0.$
14. $x-2a+b-y,\ x-2a-b+y.$
15. $x-1+a+2b,\ x-1-a-2b.$

Article 91, page 82.

3. $a^2+a+1,\ a^2-a+1.$
4. $a^2-b^2+2ab,\ a^2-b^2-2ab.$
5. $a^2+b^2+ab,\ a^2+b^2-ab,\ a^4+b^4-a^2b^2.$
6. $a^2+2a-3,\ a^2-2a-3.$
7. $x^2+x-1,\ x^2-x-1.$

Article 92, page 83.

2.	$a+b,\ c+d.$	8.	$x+2,\ x^2+4.$
3.	$a-b,\ c-d.$	10.	$3x-a,\ 2x+y.$
5.	$x-a,\ x^2+a^2.$	12.	$x-1,\ x^2+1.$
7.	$x-y,\ x+y,\ x-y.$	15.	$a+b,\ a+b,\ a+b.$

Article 93, page 85.

3.	$x+3,\ x+2.$	12.	$x-6,\ x-1.$	21.	$1+2x,\ x-1.$
4.	$x+4,\ x+3.$	13.	$x-5,\ x+1.$	22.	$2-x,\ x+1.$
5.	$x+5,\ x+1.$	14.	$x+5,\ x-1.$	23.	$x+11,\ 10-x.$
6.	$x+7,\ x+3.$	15.	$x-5,\ x+4.$	24.	$x+7a,\ x-6a.$
7.	$x+3,\ x-1.$	16.	$x+5,\ x-4.$	25.	$x+3a,\ x-2a.$
8.	$x-3,\ x+1.$	17.	$x+10,\ x-1.$	26.	$ab-5c,\ ab+2c.$
9.	$x-6,\ x+2.$	18.	$x+7,\ x-5.$	27.	$xy+4z,\ xy-3z.$
10.	$x-3,\ x-2.$	19.	$x+5a,\ x+2a.$	28.	$a,\ x-6,\ x-5.$
11.	$x-6,\ x-2.$	20.	$x+11a,\ x+a.$	29.	$x,\ x+7,\ x+2.$

Article 94, page 86.

6. $x+1,\ 3x-2.$
8. $x+2,\ 2x+1.$
9. $3x+1,\ x+3.$
10. $x+2,\ 2x-1.$
11. $x-5,\ 3x+2.$
12. $x-3,\ 5x-7.$
13. $x-7,\ 5x+3.$
14. $x+7,\ 5x+3.$
15. $3x-2,\ x-1.$
16. $3x-1,\ x-2.$
17. $x+1,\ 2x-1.$
18. $x-1,\ 2x-1.$
19. $x-1,\ 5x+2.$
20. $x+2,\ 5x-1.$
21. $x+1,\ 3x-1.$
22. $x+1,\ 7x-3.$
23. $x-3,\ 7x+1.$
24. $x+3,\ 7x-1.$
25. $x-1,\ 7x-3.$
26. $x+1,\ 7x+3.$
27. $x-1,\ 7x-2.$
28. $x-1,\ 7x+2.$

Article 96, page 88.

5. $a-b,\ a^2+ab+b^2.$
6. $x+y,\ x^2-xy+y^2.$
7. $c-1,\ c^2+c+1.$
8. $c+1,\ c^2-c+1.$
9. $x+2,\ x^2-2x+4.$
10. $1-x,\ 1+x+x^2.$
11. $x+3,\ x^2-3x+9.$
12. $x-3,\ x^2+3x+9.$
13. $x-a,\ x^4+x^3a+x^2a^2+xa^3+a^4.$
14. $x-2,\ x^4+2x^3+4x^2+8x+16.$
15. $x+4,\ x^2-4x+16.$
16. $x-4,\ x^2+4x+16.$
17. $2,\ 3,\ x-2,\ x^2+2x+4.$
18. $3a-c,\ 9a^2+3ac+c^2.$
19. $a-7,\ a^2+7a+49.$
20. $a-3,\ a^4+3a^3+9a^3+27a+81.$
21. $5-a,\ 25+5a+a^2.$
22. $a-5,\ a^2+5a+25.$
23. $5a-1,\ 25a^2+5a+1.$
24. $2a-cd,\ 4a^2+2acd+c^2d^2.$
25. $x^2-y,\ x^4+x^2y+y^2.$
26. $3y-x^2,\ 9y^2+3x^2y+x^4.$
27. $abc-d,\ a^2b^2c^2+abcd+d^2.$
28. $abc-1,\ a^2b^2c^2+abc+1.$
29. $2,\ 3a-1,\ 9a^2+3a+1.$
30. $3,\ 2a-3b,\ 4a^2+6ab+9b^2.$
31. $2,\ 2,\ x,\ 2x-3y,\ 4x^2+6xy+9y^2.$

Article 97, page 89.

4. $a^4 + b^4,\ a^2 + b^2,\ a + b,\ a - b.$

5. $x + y,\ x^2 - xy + y^2,\ x - y,\ x^2 + xy + y^2.$

6. $a^2 + 1,\ a + 1,\ a - 1.$

7. $1 + a^2,\ 1 + a,\ 1 - a.$

8. $a + 1,\ a^2 - a + 1,\ a - 1,\ a^2 + a + 1.$

9. $1 + a,\ 1 - a + a^2,\ 1 - a,\ 1 + a + a^2.$

10. $1 + a^4,\ 1 + a^2,\ 1 + a,\ 1 - a.$

11. $a^4 + 1,\ a^2 + 1,\ a + 1,\ a - 1.$

12. $a+b,\ a^4-a^3b+a^2b^2-ab^3+b^4,\ a-b,\ a^4+a^3b+a^2b^2+ab^3+b^4.$

13. $a^2 + 4,\ a + 2,\ a - 2.$

14. $a^2 + 9,\ a + 3,\ a - 3.$

15. $9 + a^2,\ 3 + a,\ 3 - a.$

16. $a^2b^2 + c^2,\ ab + c,\ ab - c.$

17. $x^2+y^2,\ x^4-x^2y^2+y^4,\ x+y,\ x^2-xy+y^2,\ x-y,\ x^2+xy+y^2.$

18. $a^7,\ a + 1,\ a^2 - a + 1,\ a - 1,\ a^2 + a + 1.$

19. $a^8,\ a^4 + 1,\ a^2 + 1,\ a + 1,\ a - 1.$

20. $a + 2,\ a^2 - 2a + 4,\ a - 2,\ a^2 + 2a + 4.$

21. $x,\ 4x^2 + 1,\ 2x + 1,\ 2x - 1.$

22. $a^6,\ a + 1,\ a^2 - a + 1,\ a - 1,\ a^2 + a + 1.$

23. $a^4b^4,\ a^4b^4 + 1,\ a^2b^2 + 1,\ ab + 1,\ ab - 1.$

24. $x^4y^4,\ xy + 1,\ x^2y^2 - xy + 1,\ xy - 1,\ x^2y^2 + xy + 1.$

Article 100, pages 90–91.

1. $a + b.$ 2. $a^2 + ab + b^2.$ 3. $a^2 - ab + b^2.$

4. $a^4 - a^3b + a^2b^2 - ab^3 + b^4.$ 5. $a^4 + a^3b + a^2b^2 + ab^3 + b^4.$

6. $-1.$ 11. $2x + 3y.$ 16. $a + b.$

7. $a - 1.$ 12. $1 + 3x.$ 17. $3 + x.$

8. $a^2 + a + 1.$ 13. $4x^2 + y^2.$ 18. $x^2 - y^2.$

9. $a^2 - a + 1.$ 14. $25 - 5a + a^2.$ 19. $9 - 3x + x^2.$

10. $1 + a + a^2.$ 15. $a - b.$ 20. $x^3 - y^3.$

21. $x + 3.$
22. $x - 3.$
23. $a + b.$
24. $a + 1.$
25. $a - b.$
26. $a - 1.$
27. $x - 1.$
28. $x + 10.$
29. $x - 3a.$
30. $x + 9y.$
31. $(x+1)(x+2).$
32. $(x-1)(x+3).$
33. $a + b.$
34. $a + b - c.$
35. $a - b.$
36. $a^2 - ab + b^2.$
37. $a + b + c.$
38. $a^8 - a^4b^4 + b^8.$
39. $a + c.$
40. $a + b.$

Article 101, page 91.

1. $x^2 - 7,\ x^2 + 6.$
2. $x - 10y,\ x + y.$
3. $x^5,\ x + 1,\ x^2 - x + 1,\ x - 1,\ x^2 + x + 1.$
4. $3x + 1,\ 3x + 1.$
5. $x - 7y,\ x - y.$
6. $x^5,\ x^4 + 1,\ x^2 + 1,\ x + 1,\ x - 1.$
7. $a,\ a + 1,\ x - 1.$
8. $x + 2,\ x^2 + 4.$
9. $x - y + a,\ x - y - a.$
10. $a - b + 1,\ a - b - 1.$
11. $4x^2 + 1,\ 2x + 1,\ 2x - 1.$
12. $2a,\ a + b,\ a^2 - ab + b^2,\ a - b,\ a^2 + ab + b^2.$
13. $x - 7,\ 7x - 1.$
14. $2,\ a,\ x,\ x - 6,\ x - 5.$
15. $a - 3,\ 2a + 5.$
16. $a + 3x,\ a + 2x.$
17. $x,\ y,\ x + y,\ x - y,\ x - y.$
18. $x + y + 1,\ x - y.$
19. $a + d + b - c,\ a + d - b + c.$
20. $a + b - c,\ a + b - c.$
21. $a + b + 1,\ a + b.$
22. $x^2 + y^2 + xy,\ x^2 + y^2 - xy,\ x^4 + y^4 - x^2y^2.$
23. $x + 2y,\ x - 2y + 1.$
24. $2 + x - y,\ 2 - x + y.$
25. $x - 1 + x^2,\ x - 1 - x^2.$
26. $x + 1,\ 2x - 1.$
27. $a,\ a^2 - a + 1,\ a^2 + a + 1.$
28. $a + 1,\ a + 1,\ a + 1.$
29. $a^2,\ a - 2b,\ a^2 + 2ab + 4b^2.$
30. $a - c,\ a^2 + ac + c^2,\ x + y,\ x - y.$

Article 105, page 93.

3.	$a^2 b$.	12.	$3a^2c^4$.	21.	$a + x$.
4.	bc.	13.	a.	22.	$x + y$.
5.	x.	14.	$a + 2b$.	23.	$x - 3$.
6.	$3bc$.	15.	$(x + a)^2$.	24.	$x^2 + ax + a^2$.
7.	abc.	16.	$3(a + b)^3$.	25.	$x^4 - a^2x^2 + a^4$.
8.	$4ac$.	17.	$2(a - b)$.	26.	$(x - y)^4$.
9.	$4(x - 3)$.	18.	$x + 4$.	27.	$x - 3$.
10.	$x + 1$.	19.	$x + 7$.	28.	$3x + 1$.
11.	$x + 3$.	20.	$x^2 - y^2$.		

Article 106, page 96.

6.	$8x^2 + 14x - 15$.	13.	$x + 3$.	23.	$a(a + b)$.
7.	$x - 5$.	15.	$2(x^2 - 2y)$.	24.	$x(2x - 1)$.
10.	$x - y$.	18.	$x(x - 1)$.	25.	$x - 3$.
11.	$2x - 1$.	21.	$x^2 - 2x + 4$.		

Article 110, page 98.

3.	$2^2 \cdot 3a^9$.	8.	a^2yz^2.	13.	$a^2b^2(x - y)$.
4.	$2 \cdot 3 \cdot 5x^2y^2$.	9.	$3x(a - x)$.	14.	$a^3b^2(x^2 - a^2)$.
5.	abc.	10.	$abc(a - c)$.	15.	$abc(x + a)$.
6.	$2^2 \cdot 3 \cdot 5x^4y^2$.	11.	$2^2a^2x(a + x)$.	16.	$(x + y)(x - y)^2$.
7.	$2^2 \cdot 3x^4y^4$.	12.	$3 \cdot 7(a + b)$.	17.	$2^2(x^3 - y^3)$.

18.	$x(x^2 - 1)$.	25.	$(x - a)^2(x - b)$.
19.	$(x + a)^2(x - a)^2$.	26.	$(x - 2)^2(x^2 + 2x + 4)$.
20.	$2^3 \cdot 3(x + 1)^3(x - 1)$.	27.	$(x - y)^3(x^2 + xy + y^2)$.
21.	$x(x^2 - y^2)$.	28.	$x^6 - y^6$.
22.	$x^4 - y^4$.	29.	$(x + 3)^3(x^2 - 3x + 9)$.
23.	$x^2 - 9$.	30.	$(x^3 + y^3)(x - y)$.
24.	$a^2x^2 - b^3x^2$.	31.	$(x^3 - y^3)(x^3 + y^3)$.

Article 114, page 101.

2. $-\dfrac{-3a}{x-y}, -\dfrac{3a}{y-x}, \dfrac{-3a}{y-x}$. 3. $-\dfrac{y-x}{z}, -\dfrac{x-y}{-z}, \dfrac{y-x}{-z}$.

4. $-\dfrac{-5x}{a^2-b^2}, -\dfrac{5x}{b^2-a^2}, \dfrac{-5x}{b^2-a^2}$.

5. $-\dfrac{1-x^2}{x^2-x-2}, -\dfrac{x^2-1}{x-x^2+2}, \dfrac{1-x^2}{x-x^2+2}$.

6. $-\dfrac{a-b}{y-x}, -\dfrac{b-a}{x-y}, \dfrac{a-b}{x-y}$.

7. $-\dfrac{1-3x}{1-x^2}, -\dfrac{3x-1}{x^2-1}, \dfrac{1-3x}{x^2-1}$.

10. $\dfrac{x-z}{y}$. 12. $\dfrac{x-a}{y-b}$. 14. $\dfrac{y-z}{x(x^2-1)}$.

11. $\dfrac{x+y-a}{(b-a)(c-d)}$. 13. $\dfrac{x+y-a}{9-z^2}$.

Article 116, pages 102, 103.

3. $\dfrac{ac}{b}$. 8. $\dfrac{b}{2a+3c}$ 13. $\dfrac{-1}{x+y}$.

4. $\dfrac{b^2}{4x^3}$. 9. $\dfrac{a}{b}$. 14. $\dfrac{b-a}{a+b}$.

5. $\dfrac{b^2}{3x}$. 10. $\dfrac{a-b}{a+b}$. 15. $\dfrac{-1}{2+x}$.

6. $\dfrac{a}{y-1}$. 11. $\dfrac{a-b}{a^2-ab+b^2}$. 16. $\dfrac{x}{x-4y}$.

7. $\dfrac{a-b}{ab}$. 12. $\dfrac{a^2+ab+b^2}{a+b}$.

17. $\dfrac{x^2+1}{(x^2-x+1)(x^2+x+1)}$. 18. $\dfrac{a-x}{a+x}$.

19. $\dfrac{2x}{1+2x}$. 20. $\dfrac{-x}{x+4a}$. 21. $\dfrac{a+x}{a(a-x)}$.

22. $\dfrac{5a}{x+3a}$.

23. $\dfrac{x-2}{x-4}$.

24. $\dfrac{x+2y}{x-2y}$.

25. $\dfrac{x+2}{x^2-2x+4}$.

26. $\dfrac{a+3}{3a-4}$.

27. $\dfrac{a^2+b^2}{a^2+ab+b^2}$.

28. $\dfrac{a-b-c}{a-b+c}$.

29. $\dfrac{1}{a-b}$.

30. $\dfrac{x-y}{x+1}$.

Article 117, page 105.

2. $a+b+\dfrac{c}{a}$.

3. $a-b+\dfrac{b^2}{a}$.

4. $2a-4b-\dfrac{c}{2x}$.

5. $x+2-\dfrac{7}{x+2}$.

6. a^2+ab+b^2.

7. $a^2-a+1-\dfrac{2}{a+1}$.

8. $1+a+a^2+\dfrac{a^3}{1-a}$.

9. $1-a+a^2$.

10. $x^2+\dfrac{1}{x^2+x+1}$.

11. $a^4+a^3b+a^2b^2+ab^3+b^4$.

12. a^2+b^2-ab.

Article 118, page 106, 107.

4. $\dfrac{ac+b}{c}$.

5. $\dfrac{x+1}{x}$.

6. $\dfrac{x-1}{x}$.

7. $\dfrac{xy+1}{y}$.

8. $\dfrac{xy-1}{y}$.

9. $\dfrac{a-bc}{b}$.

10. $\dfrac{2-x}{1-x}$.

11. $\dfrac{x}{1-x}$.

12. $\dfrac{2b}{a-b}$.

13. $\dfrac{a^2}{a+x}$.

14. $\dfrac{x^2}{x+y}$.

15. $\dfrac{a^2+ab+b^2}{a-b}$.

16. $\dfrac{a^2-9b^2+9b}{a-3b}$.

17. $\dfrac{2}{1-x}$.

18. $\dfrac{a^2-ab}{b}$.

Article 119, page 109.

4. $\frac{3a}{12}, \frac{2b}{12}.$

5. $\frac{8x}{12}, \frac{2x}{12}, \frac{3x}{12}.$

6. $\frac{8ac}{10bc}, \frac{3ab}{10bc}.$

7. $\frac{2az}{6xyz}, \frac{b}{6xyz}, \frac{3cx}{6xyz}.$

8. $\frac{x^2}{xy}, \frac{y^2}{xy}, \frac{3x^2y}{xy}.$

9. $\frac{a^2}{abc}, \frac{b^2}{abc}, \frac{abc}{abc}.$

10. $\frac{3x-3}{6}, \frac{2x-4}{6}, \frac{x-3}{6}.$

11. $\frac{ac-bc}{abc}, \frac{ab-bc}{abc}, \frac{ab-ac}{abc}.$

12. $\frac{a+b}{a^2-b^2}, \frac{1}{a^2-b^2}.$

13. $\frac{x-2}{(x-2)^2}, \frac{2}{(x-2)^2}.$

14. $\frac{x+y}{x^2-y^2}, \frac{2}{x^2-y^2}.$

15. $\frac{xy}{25x^2-y^2}, \frac{5x^2-xy}{25x^2-y^2}.$

16. $\frac{xy-y^2}{xy(x^2-y^2)}, \frac{x^2+xy}{xy(x^2-y^2)}.$

17. $\frac{x^2y^2+y^4}{xy(x^4-y^4)}, \frac{x^4-x^2y^2}{xy(x^4-y^4)}.$

18. $\frac{2a^2+2ab}{8(a^2-b^2)}, \frac{b}{8(a^2-b^2)}.$

19. $\frac{5-10x}{1-4x^2}, \frac{6+12x}{1-4x^2}, \frac{2-3x}{1-4x^2}.$

20. $\frac{3abc+3b^2c}{6abc(a^2-b^2)}, \frac{4ac}{6abc(a^2-b^2)}, \frac{3a^2b-3ab^2}{6abc(a^2-b^2)}.$

Article 120, pages 110, 111.

4. $\frac{3x-4}{6}.$

5. $\frac{x+21}{6}.$

6. $\frac{9-x}{12}.$

7. $\frac{a+b}{ab}.$

8. $\frac{a+b}{abx}.$

9. $\frac{a(x+a)}{x^2}.$

10. $\dfrac{x(b-a)}{ab}$.

11. $\dfrac{(a+x)^2}{a^2x^2}$.

12. $\dfrac{11a}{6x}$.

14. $\dfrac{2a}{a^2-b^2}$.

15. $\dfrac{2y}{x^2-y^2}$.

16. $\dfrac{2x^2}{x^2-a^2}$.

17. $\dfrac{2ax}{a^2-x^2}$.

18. $\dfrac{x+1}{x(x-1)}$.

19. $\dfrac{(x-y)^2}{4y(x+y)}$.

20. $\dfrac{(a+b)^2}{4a(a-b)}$.

21. $\dfrac{a+x}{x(a-x)}$.

22. $\dfrac{a+x}{ax}$.

23. $\dfrac{x^2-x+1}{(x-1)^2}$.

24. $\dfrac{2}{a-b}$.

25. $\dfrac{2(a^2+ax-x^2)}{a^2-x^2}$.

26. $\dfrac{a^2}{a^2-x^2}$.

27. $\dfrac{-2}{a+b}$.

Article 121, page 113.

7. $\dfrac{2x}{a}$.

8. $\dfrac{3z}{8xy}$.

9. $\dfrac{2x^4}{a^8}$.

10. $\dfrac{5a^2}{3x}$.

11. $\dfrac{x}{x-3}$.

12. $\dfrac{x^2}{y(x+y)}$.

13. $\dfrac{x}{a+x}$.

14. $\dfrac{x^2}{y(x+y)}$.

15. $\dfrac{a^2-ax+x^2}{(a+x)(a-x)^2}$.

16. $\dfrac{x^2+xy+y^2}{x+y}$.

17. $\dfrac{a}{b}$.

Article 123, pages 115, 116.

2. 10, 4.
3. 16, 9.
4. 29, 21.
5. 60, 40.
6. 15, 3.
7. 19, 10.
8. 60, 45.
9. 30, 23.
10. $x+y$, $x-y$.
11. $4x^2-y$, $2x^2-2y$.
12. $4a-c$, $-a-5c$.
13. 12, 8.
16. $2\frac{2}{5}$.
17. $4\frac{2}{17}$.

Article 124, page 117.

19. $p = \dfrac{i}{t\,r}$. Divide the interest by the product of the time and the rate.

20. $p = \dfrac{a}{1 + t\,r}$. Divide the amount by 1 plus the product of the time and the rate.

21. $r = \dfrac{i}{p\,t}$. Divide the interest by the product of the principal and the time.

22. $r = \dfrac{a - p}{p\,t}$. From the amount subtract the principal, and divide the remainder by the product of the principal and the time.

23. 3 years.	24. 6 years.	25. 7 %.
26. \$1000.	27. $16\frac{2}{3}$ years. 20 years. 25 years.	

Article 125, page 118.

28. \$250.	31. \$985.22 +.	34. \$4.31 +.
29. \$500.	32. \$36.	35. \$4.95 +.
30. \$400.	33. \$72.	36. \$9.90 +.

Pages 119–128.

1. 4.	12. $\frac{28}{13}$.	23. 12.	34. 6.
2. $\frac{3}{4}$.	13. 1.	24. 5.	35. 4.
3. $\frac{1}{13}$.	14. 11.	25. 7.	36. 7.
4. − 2.	15. − 3.	26. 3.	38. 2.
5. − 3.	16. 7.	27. 7.	39. 3.
6. 0.	17. $\frac{2}{9}$.	28. 65.	40. $66\frac{2}{3}$.
7. 10.	18. 6.	29. 3.	41. 20.
8. 6.	19. 15.	30. 19.	42. 20.
9. − 21.	20. 7.	31. 21.	43. 5.
10. 2.	21. $3\frac{1}{7}$.	32. $\frac{4}{5}$.	44. 10.
11. 5.	22. 5.	33. 4.	45. 5.

46. $2b$.

47. $a+b$.

48. 1.

49. a^2+ab+b^2.

50. 1.

51. $a+b$.

53. $\dfrac{ab+b-a}{2a-b}$.

54. $a+b+c$.

55. $\dfrac{ab}{a+b}$.

56. ab.

57. $\dfrac{6a^2-5b-6b^2}{6a-b}$.

58. $2a-b$.

59. $\dfrac{a-1}{b}$.

60. $b-1$.

61. $\dfrac{b}{c}$.

62. bm.

63. 1.

64. $\dfrac{2a^3}{b-1}$.

65. 66, 35.

66. 4.

67. $7\frac{1}{2}$ hours.

68. \$480.

69. A, \$450; B, \$180; C, \$140.

71. \$7200.

72. A, \$760; B, \$880; D, \$520.

73. 85, 35.

74. 210 miles.

78. $\dfrac{abc}{ab+ac+bc}$

79. 7.

80. $\frac{1}{2}$.

81. 12.

82. 7.

83. 9.

84. $-1\frac{1}{2}$.

85. 3.

86. 1.

87. -2.

88. $6\frac{1}{5}$.

89. $\dfrac{a+b}{2}$

90. $\dfrac{2ab}{a+b}$.

91. $\dfrac{ab}{b-a}$.

92. $\begin{cases} x=5. \\ y=10. \end{cases}$

93. $\begin{cases} x=-2. \\ y=-1. \end{cases}$

94. $\begin{cases} x=4. \\ y=-3. \end{cases}$

95. $\begin{cases} x=\frac{5}{6}. \\ y=-\frac{5}{6}. \end{cases}$

96. $\begin{cases} x=3. \\ y=6. \end{cases}$

97. $\begin{cases} x=\frac{1}{2}. \\ y=\frac{1}{6}. \end{cases}$

98. $\begin{cases} x=2. \\ y=3. \end{cases}$

99. $\begin{cases} x=\frac{1}{5}. \\ y=1. \end{cases}$

100. $\begin{cases} x=\dfrac{a+b}{2}. \\ y=\dfrac{a-b}{2}. \end{cases}$

101. $\begin{cases} x=\dfrac{cn-bd}{an-bm}. \\ y=\dfrac{cm-ad}{bm-an} \end{cases}$

102. $\begin{cases} x=\dfrac{ab}{a+b}. \\ y=\dfrac{ab}{a+b} \end{cases}$

103. $\begin{cases} x=\dfrac{a^2-b^2}{am-bn} \\ y=\dfrac{a^2-b^2}{an-bm}. \end{cases}$

104. $\begin{cases} x = 1. \\ y = 2. \\ z = 3. \end{cases}$

105. $\begin{cases} x = 5. \\ y = 3. \\ z = 2. \end{cases}$

106. $\begin{cases} x = 2. \\ y = \frac{1}{2}. \\ z = -2\frac{1}{2}. \end{cases}$

107. $\begin{cases} x = \frac{1}{2}(a - b + c). \\ y = \frac{1}{2}(a + b - c). \\ z = \frac{1}{2}(b - a + c). \end{cases}$

108. $\begin{cases} x = \dfrac{1}{2a}. \\ y = \dfrac{1}{2b}. \\ z = \dfrac{1}{2c}. \end{cases}$

109. $\begin{cases} x = \frac{2}{3}. \\ y = 2. \\ z = \frac{2}{5}. \end{cases}$

110. $1600.

111. 26.

112. 15.

114. $\begin{cases} \text{Eagles, 162.} \\ \text{Dollars, 105.} \end{cases}$

115. $\begin{cases} \text{14 acres.} \\ \text{10 acres.} \end{cases}$

117. $5500.

118. 84 years.

119. 19.

123. 52 years.

124. $\begin{cases} \text{3 years 6 months after} \\ \quad \text{the end of the 7 mos.} \end{cases}$

125. $\begin{cases} \text{1st, 120 liters.} \\ \text{2d, } 93\frac{1}{3} \text{ liters.} \\ \text{3d, 70 liters.} \end{cases}$

129. $\begin{cases} \text{Coffee, \$0.35.} \\ \text{Sugar, \$0.08.} \end{cases}$

131. $1\frac{1}{2}$ and 1.

133. $\begin{cases} \text{1st, 720 pounds} \\ \text{2d, 512 pounds.} \end{cases}$

134. $\begin{cases} \text{1st, 25 days at \$0.80.} \\ \text{2d, 20 days at \$0.60.} \end{cases}$

135. $\begin{cases} \text{Woollen, \$1.25.} \\ \text{Silk, \$0.75.} \end{cases}$

136. $\begin{cases} \text{29 lines.} \\ \text{32 letters.} \end{cases}$